AN ELEMENTARY TREATISE

ON

STATICALLY INDETERMINATE STRESSES

AN ELEMENTARY TREATISE
ON
STATICALLY INDETERMINATE STRESSES

BY

JOHN IRA PARCEL, A.B., B.S. C.E.,

Professor of Structural Engineering
University of Minnesota

AND

GEORGE ALFRED MANEY, M.S., C.E.,

Assistant Professor of Structural Engineering
University of Minnesota

NEW YORK
JOHN WILEY & SONS, Inc.
London: CHAPMAN & HALL, Limited
1926

Printed in U. S. A.

PRESS OF
BRAUNWORTH & CO.
BOOK MANUFACTURERS
BROOKLYN. NEW YORK

PREFACE

This book has grown out of the authors' needs in teaching the subject of Indeterminate Structures during the past fifteen years. It is intended to present as clearly as possible, and as fully as is consistent with an elementary treatise, the fundamental methods of attack on the problem of indeterminate stresses, and to illustrate these methods by application to some of the more common types of indeterminate structures.

It is believed that the book will be suitable for brief introductory courses and that it also contains sufficient material, if supplemented by some reference reading, for the longer courses now offered to advanced seniors and graduate students in many technical schools. While written primarily as a class room text, it is hoped that the book will prove useful to engineers wishing to work up the subject by independent study.

Some brief remarks on the general plan of the work may not be out of place.

Chapters I–III, comprising more than one-third of the book, are devoted to an exposition of the theory of elastic deflections and to a broad treatment of the general problem of indeterminate stresses. Every effort is made to show the essential unity of the subject underlying the great diversity in method.

Chapters IV–VII treat specifically the continuous girder, the rigid frame, the elastic arch, and secondary stresses. With few exceptions, the treatment is devoted entirely to the development and illustration of methods of analysis.

Chapter VIII, containing a general discussion and historical survey, is in the nature of an appendix. It is hoped that it may stimulate the reader's interest in some of the broader phases of the subject.

Among the special features of the work, in addition to those just noted, may be mentioned the unusually full treatment of the rigid frame (which has grown so rapidly in importance of late), the wide use made of the slope deflection method, and the large number of numerical problems accompanying the text.

To keep within the limits of a moderate-sized volume it was necessary to exclude some important topics which might well claim a place even in an elementary treatise. Among these may be mentioned the

theory of suspension systems, of wind stresses in tall building frames, and the graphic treatment of continuous girders, frames and arches. Whether in all cases the selection has been wise must be left to the judgment of professional colleagues who use the book.

In any such book as this, the indebtedness to other works on the subject is of course very great. It was the intent of the authors to give all sources of specific information in the footnotes; for any cases where they may have failed to do this, they wish to make acknowledgment here. They are under especial obligation to Lieutenant Joseph A. Wise, formerly Instructor in Structural Engineering, and to Messrs. Donald O. Nelson and Frank E. Nichol, Fellows in Structural Engineering of the University of Minnesota, for important assistance in the preparation of the manuscript. They are indebted to Mr. Gilbert C. Staehle, Consulting Engineer of Minneapolis, for some of the problems in Chapter V, and to Professor Frank H. Constant of Princeton University and Professor Hardy Cross of the University of Illinois for most valuable criticisms and suggestions. For these services the authors wish to express their deep appreciation and thanks.

Thanks are also due Dean F. E. Turneaure and the McGraw-Hill Book Co. for permission to reproduce Figures 106 and 107*f*, respectively.

The authors can hardly hope that a book containing so much detail will be entirely free from errors, and they will greatly appreciate having these brought to their attention.

John I. Parcel
George A. Maney

University of Minnesota,
April, 1926.

CONTENTS

INTRODUCTION

A. Nature of Statical Indetermination

B. Types of Statically Indeterminate Structures

CHAPTER I

DEFLECTIONS

SECTION I—DEFLECTIONS BY METHOD OF WORK

A. Deflections by Maxwell-Mohr Method (Dummy Unit Loading)

B. Deflection as the Partial Derivative of the Internal Work of Deformation

SECTION III—INFLUENCE LINES FOR STATICALLY INDETERMINATE STRUCTURES

SECTION IV—THE METHOD OF LEAST WORK

SECTION V—TEMPERATURE AND OTHER NON-ELASTIC EFFECTS

SECTION VI—GENERAL SUMMARY

CHAPTER III

SPECIAL METHODS OF ATTACK

SECTION I—MOMENT AREAS AND ELASTIC WEIGHTS APPLIED TO EVALUATION OF DEFLECTIONS

SECTION II—SPECIAL SELECTION OF BASIC STRUCTURE

SECTION III—THE DIRECT APPLICATION OF THE MOMENT AREA PRINCIPLE

SECTION IV—THE SLOPE-DEFLECTION METHOD

CHAPTER IV

CONTINUOUS GIRDERS

CHAPTER V

THE RIGID FRAME

CHAPTER VI

THE ELASTIC ARCH

SECTION I—THE TWO-HINGED ARCH

A. The Arch Rib

B. The Spandrel Braced Arch

SECTION II—THE HINGELESS ARCH RIB

CHAPTER VII

SECONDARY STRESSES

CHAPTER VIII

GENERAL DISCUSSION OF STATICALLY INDETERMINATE CONSTRUCTION—HISTORICAL REVIEW—BIBLIOGRAPHY

A. General Discussion

B. Historical Review

AN ELEMENTARY TREATISE ON STATICALLY INDETERMINATE STRESSES

INTRODUCTION

A. Nature of Statical Indetermination

1. Definition.—Any structure in which the reactions or stresses are not fully defined, in terms of known quantities, by the necessary relations of static equilibrium, is said to be "statically indeterminate."

For the student who is unfamiliar with the conception, some elaboration of this definition may be helpful.

2. Structural Stability.—First we may recall some facts in the fundamental theory of simple structures. The prime requisite in any structure, as an engineer views it, is *stability*. The bridge must maintain its roadway at a prescribed level; the steel skeleton of an office building must hold the walls and floors rigidly in place; the dam or retaining wall must keep a fixed position against the pressure of water or earth. We specify, therefore, in all structures, that the structure as a whole and *all its parts* shall satisfy the conditions of static equilibrium. These conditions are but three in number and are expressed mathematically by the well-known equations: *

$$\Sigma F_x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$\Sigma F_y = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$\Sigma M = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

3. Examples.—We may note three cases (see Figs. 1, 2 and 3).

Fig. 1 is obviously unstable. Unless the load P acts along the line AB, the structure cannot maintain its position no matter how strong

* It is to be understood here and throughout this book that we are dealing only with forces lying in a plane.

the member AB nor how firmly supported. Fig. 2 is clearly a stable form for all conditions of loading and the simplest form possible for maintaining the point A in a fixed position against the action of any force P. The point cannot move appreciably except by the failure of one of the bars.

It should be clear from the above that any pair of bars in Fig. 3 will constitute a stable system, and therefore this structure has one superfluous member. Fig. 1 is essentially unstable; Fig. 2 is "just stable";

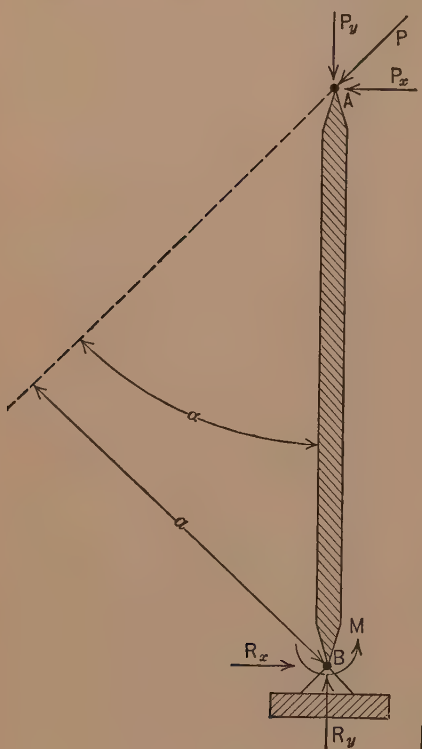


FIG. 1

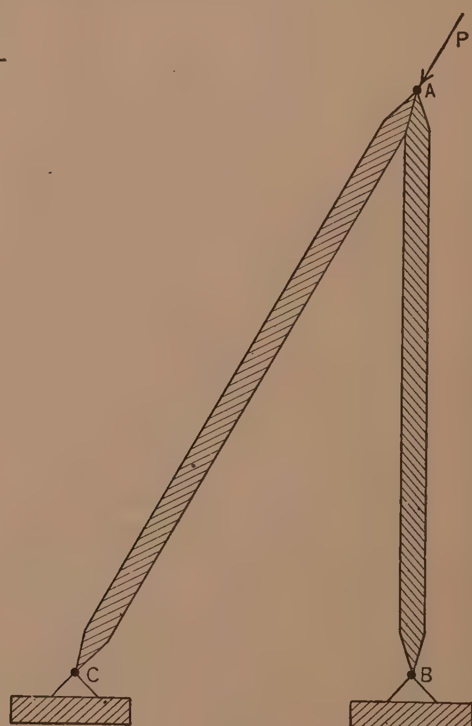


FIG. 2

Fig. 3 is "over-stable." Or, to put it another way, (1) is structurally defective, (2) is structurally sufficient, (3) is structurally redundant.

4. Analytical Conditions.—Most solutions of practical structural problems involve an answer to the question—"Given a structure and a loading, what must be the value of a given reaction or stress in a given member to insure equilibrium?" Applying this method to Fig. 1 we see that at B we require $R_x = P_x$, $R_y = P_y$ and $M = P \cdot a$. But, from the conditions of the problem (smooth pin at B), we cannot develop

a moment at the point of support, i.e., we have more conditions than means of satisfying them. Algebraically, we say there are more equations than there are unknowns, and no solution, in general, can exist.

In the familiar case of Fig. 2, known methods of stress analysis show that for any condition of loading there is one and only one set of values of reactions and stresses which are consistent with equilibrium. Algebraically, we have exactly the same number of unknowns as we have equations of condition.

Turning to Fig. 3 we note that at joint A, for instance, we may

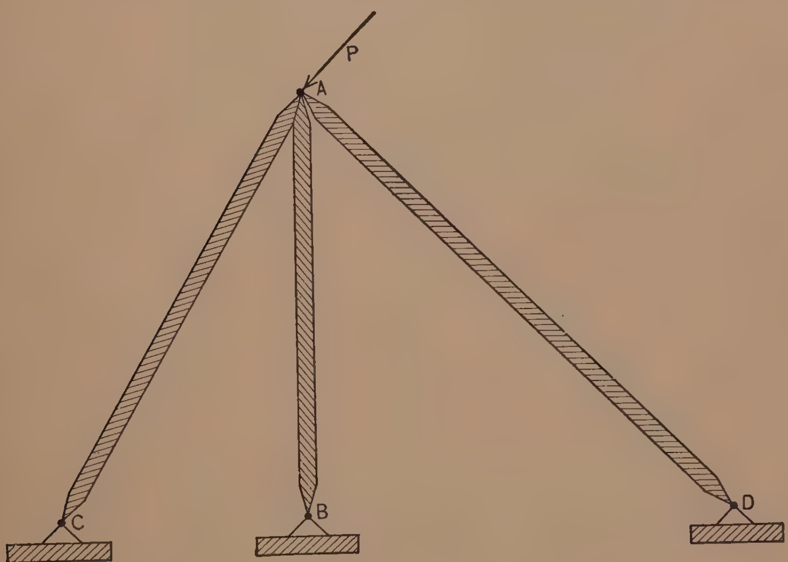


FIG. 3

remove any one of the bars and yet satisfy all the requirements for equilibrium by suitable stresses in the other two. If we arbitrarily assume *any* value for AD (say $\frac{P}{3}, \frac{P}{2}$ or similar value), we at once find the proper equilibrating values for the other bars by application of equations (a) and (b). That is to say, there is more than one set of values (actually an indefinite number) of the reactions and stresses in the structure of Fig. 3 which will completely satisfy the requirements of equilibrium,—more unknowns to determine than equations of condition, and no definite solution can be effected.

We may say, then, that a solution of (1) is in general impossible, the solution of (2) is definite and unique, the solution of (3) is indeterminate —*statically* indeterminate, we should say, because thus far only statical relations have been invoked in the solution.

5. Generality of Statical Requirements.—The following point cannot be too strongly emphasized: To say that the three statical equations are insufficient for the solution of a framework of the type of (3) does not mean that they do not apply with all the force they do in any case. *Any* useful structure must fully conform to the laws of static equilibrium.* In some structures these laws, mathematically expressed,

Fig. 4

suffice for a complete analysis of stresses; in others they do not; but this in nowise relieves the latter of the fundamental requirements.

6. Principle of Consistent Distortions.—Seeing that the laws of equilibrium alone do not define the reactions and stresses in certain structures, we naturally ask what are the conditions which do serve to define these quantities. We know that the stresses and reactions are

* This statement is subject to obvious qualification—movable bridges, fixed bridges subjected to suddenly applied live loads, cranes, ships, etc., may not, in a sense, fully conform to statical conditions, but this discrepancy is of no great importance so far as methods of analysis are concerned.

not arbitrary and lawless; "real indeterminateness does not exist in nature." * To find the answer to this question we must undertake a more exact inquiry into the behavior of a structure under stress. Many problems in stresses can be analyzed quite correctly on the assumption that the structure is a rigid body; but, of course, all bodies of which we have any knowledge are actually at least slightly deformable and the deformations and the corresponding stresses are connected by very definite experimental laws, as the student has already learned from the study of mechanics of materials. Without taking up the matter in detail at this stage, it is not difficult to see how this fact affects the problem under consideration. Take the simplest possible case, as shown in Fig. 4. If we arbitrarily assume, for example, that $AB = 0$, we arrive at a set of values for the remaining stresses and reactions which satisfy the laws of equilibrium and, so far as this requirement goes, are as valid as any other. Let us now examine the elastic deformations. If the bars are all equal, the point A will move slightly downward along AB because of the elastic yield of the structure CAD. But, since the three bars are rigidly attached at A, this cannot happen without inducing a considerable deformation and hence a considerable stress (actually more than in either of the other bars) in AB, which was assumed to be zero. Similarly, any other *arbitrary* set of stress values, even though complying with all conditions of equilibrium, will result in incompatible deformations. Without attempting here to justify it fully, we may now enunciate the principle upon which the answer to the preceding question is based. The reactions and stresses in any structure must not only accord with the requirements of static equi-

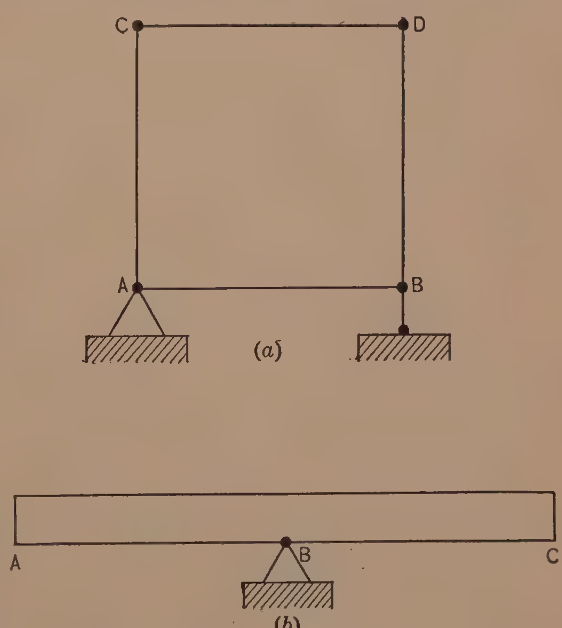


FIG. 5

* Kelvin and Tait, "Natural Philosophy," Vol. II, page 161.

librium, but *they must result in consistent elastic distortions*. The theory of statically indeterminate stresses as presented in this book consists in developing in some detail the implications of this principle in its various phases and applications.

7. Scope of Principle.—Though this law applies with all the force and generality of the laws of equilibrium, it has no significance for the analysis of structures unless they have redundant supports or members. For in all simple (i.e. "just-stable") structures the distortions, so long as they remain small, are independent of each other—any member may

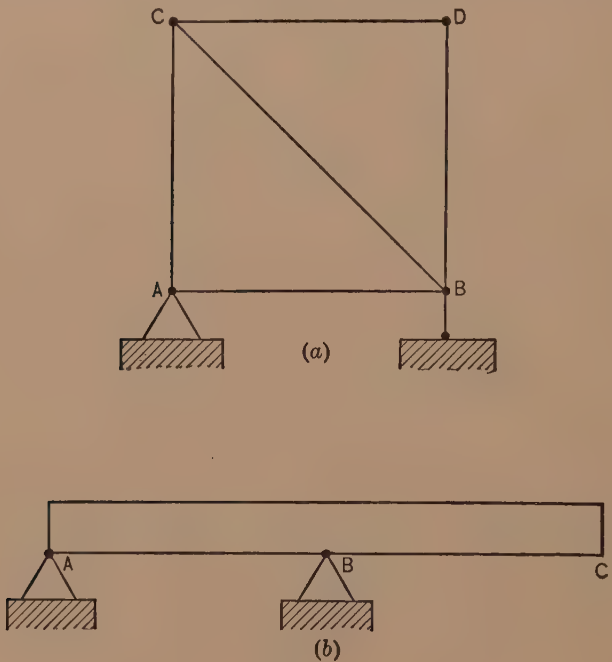


FIG. 6

change its length or any support may be displaced without thereby stressing the other parts. This must follow from the very fact that the structure has just enough members and supports for stability and no more. A little consideration should make this clear.

(*a*) and (*b*) of Fig. 5 are unstable forms. Within certain limits they may be displaced at will without awakening any resisting forces. In Fig. 6 the preceding forms have been rendered stable. From the previous discussion it is clear that, since the removal of any member of (*a*) and any support of (*b*) will result in an unstable form, therefore any member of (*a*) may change its length and either support of (*b*) may

be displaced without bringing into play any resisting forces. The law of consistent distortions has no meaning for such structures because *any set of small deformations or displacements are self-consistent.*

In Fig. 7 the forms have been made redundant. It is clear that, since the removal of any member from (*a*) or any support from (*b*) still leaves a stable (rigid) structure, therefore the deformation of any member or displacement of any support will necessarily arouse resisting forces. Viewing the problem in another way, we may say that in

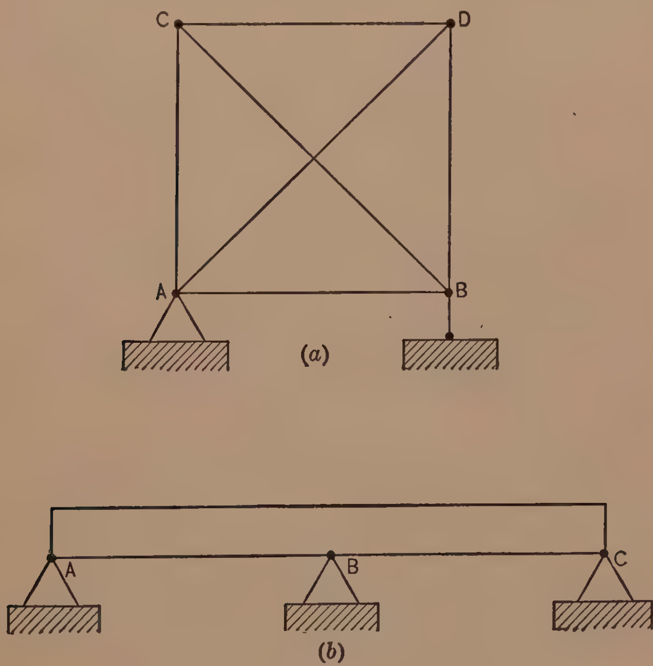


FIG. 7

(*a*) of Fig. 7, the length of any member is a function of the other lengths, while in (*a*) of Fig. 6, the lengths are (within certain limits) quite unrelated. Analogous relations hold for the supports.

B. TYPES OF STATICALLY INDETERMINATE STRUCTURES

8. Some of the more important structural problems requiring the theory of statically indeterminate stresses for their solution are tabulated below. The classification as arranged is merely for convenience of treatment; it is in no sense rigid or final, nor does the list pretend to be complete.

I. *The Continuous Girder.*

a. Ordinary restrained and continuous beams.

b. Swing bridges and turntables.

c. Continuous trusses.

Queensboro bridge (a continuous cantilever), the Scioto-ville bridge, the B. & L. E. bridge over Allegheny River at Pittsburgh, and similar types.

II. *The Elastic Arch.*

a. The two-hinged arch.

1. Solid arch rib (steel or concrete).
2. Arch truss (Hell Gate bridge).
3. Spandrel braced arch (Grand Trunk R.R. Niagara Bridge).

b. One-hinged arch (very rare).

c. Hingeless arch.

1. Solid steel rib.
2. Trussed steel rib (Eads Bridge).
3. Reinforced concrete arch (nearly all concrete arches in America are hingeless).

III. *Suspension Systems.*

a. Braced cable (Proposed Hudson River bridge).

b. Wire cable with stiffening truss (Manhattan and Williamsburg bridges).

IV. *Trusses with Redundant Members.*

Double triangular and Whipple trusses and other similar types.

V. *Rigid Frames.*

a. Simple quadrangular frames.

1. Beam and column frames in building.
2. Solid portals.
3. Box culverts.

b. Irregular frames.

1. Sewer sections and water conduits.
2. Ship frames.
3. Miscellaneous types.

VI. *Composite Frameworks.* (Beam and truss combinations.)

a. Framed bent and framed portal.

b. King and Queen post trusses.

c. Miscellaneous types.

VII. *Multiple Rigid Joint Problems.*

a. Secondary stresses in bridges.
b. Wind stresses in high building frames.
c. Open web (Vierendeel) girders.

VIII. *Girders on Continuous Yielding Supports.*

a. Railroad rail.
b. Footings and foundations.
c. Pontoon bridges.
d. Ships.

IX. *Flat Slabs, Arch Dams, Solid Domes.*

X. *Buckling of Columns, Struts, and Girder Webs.*

Many of these problems are beyond the scope of an elementary treatise. Some of them, notably the last two, involve a relatively exact investigation of the stress-strain relations within an elastic solid, and hence require the methods of the mathematical theory of elasticity for solution. This analysis differs so markedly in form from the ordinary methods of attack in statically indeterminate structures, that such problems are usually placed in a group by themselves.

9. Structures Indeterminate Internally and Externally.—We distinguish between structures that are indeterminate as to the supporting reactions and those indeterminate as to internal stresses. The former are said to be statically indeterminate externally, and the latter statically indeterminate internally.

10. Criterion of Statical Indetermination.—*Degree of Indeterminateness.* A structure which is indeterminate externally will generally be noted on inspection. Exceptional cases may arise, but they are rarely of any practical importance. The question of whether or not a framework is redundant internally is less easy to settle by inspection. The following simple criterion will suffice for all cases of plane structures likely to arise in practice. Nearly all such trusses are essentially assemblages of triangles. We may imagine them constructed by successive addition of the various joints, starting with any triangular frame as a base. Now, for stability, it is in general necessary and sufficient that each added joint shall be connected to the framework by *two* bars. Thus if n = the number of joints and m = the number of bars, $m - 3 = 2(n - 3)$, or $m = 2n - 3$.*

* In certain limiting cases a frame will be defective even though there be $2n$-3 bars; in other special cases a stable frame may be devised which will possess fewer than $2n$-3 members. As noted above, however, these frames have little practical significance.

We may approach the question from a slightly different standpoint. The student will recall, from the theory of stresses in simple structures, that for every joint of a simple truss we may write two and only two independent equations. From statical conditions alone, then, we have a total of $2n$ equations for the entire structure. Now, in general, to assure stability of the structure as a whole under any given set of forces, we must have at our disposal the magnitude and direction of one reaction and the magnitude of the other—three unknown quantities. The total number of unknowns is then $m + 3$, and if the structure is to be statically determined, this must not exceed $2n$. If it is less, then the structure is unstable. Hence a determinate and stable framework should have $m = 2n - 3$.

CHAPTER I

DEFLECTIONS

1. General.—The discussion in the preceding pages has shown that a solution of the problem of statically indeterminate stresses must be based on the elastic deflections of the structure. Indeed, it was there stated that the problem of determining the statically indeterminate forces was essentially that of so adjusting these forces as to secure consistent elastic distortions. It is evident, therefore, that a thorough study of the character of such distortions and of the methods of computing them must precede the study of indeterminate stresses.

There are also many cases where a knowledge of deflections is desirable for other reasons. For example, it is frequently desirable to camber long-span bridge trusses in such a manner that the loaded chord will take a horizontal position under maximum loading or some specified combination of dead and live loading. This means that in the unstressed state the chord will have a slight upward curvature. This result may be secured by making each top chord member a trifle longer than would correspond to the final form of the truss, a method common in ordinary cases, or by modifying the length of each member by the amount it will deform under maximum stress, a more correct method, and one preferable for very large structures. In either case, it is evident that for a rational solution of the problem it is necessary to know the relation that exists between a small change in length of any member and the corresponding displacement of any joint. A problem illustrating both cases is given on page 88.

In many erection problems, especially in the cantilever erection of long-span bridges, a knowledge of elastic deflections is of great importance. In the erection of the Sciotoville two-span continuous bridge for example, one of the spans was erected on false-work and the other cantilevered out from this to its abutment, and later jacked up to allow the end shoe to be placed. Obviously it was of the greatest importance to know beforehand what the dead load deflection of the end would be, and what jacking force would be required to lift it sufficiently to set the shoe.

In the same structure it was decided, in order to avoid high secondary

stresses, to erect the truss under considerable initial strain in the opposite direction from that developed under full loading. This process necessarily required a careful and detailed study of deflections.

Many other examples might be cited to show that it is often necessary or desirable to determine elastic deflections for their own sake. In spite of the importance of the theory of deflections in this connection, however, it still remains true that this theory finds its chief application in the analysis of statically indeterminate stresses.

We shall treat in this chapter several methods for obtaining the deflections of structures. One or two general remarks should precede this discussion. To avoid needless repetition, it should be emphasized here that in this treatise we shall deal only with deformations and displacements that are *very small* as compared with the dimensions of the structures concerned. This assumption is implicitly involved in the ordinary theory of beams and trusses, since it is there assumed that the same dimensions may be used in the strained state as in the unstrained state of the structure. For all ordinary cases, the facts fully justify the assumption. For example, the unit deformation of steel or concrete for maximum allowable working loads will seldom exceed 1 in 2000. The temperature change for a range of 100 per cent is but little more. (Coefficient of expansion for both steel and concrete is about .0000065 per degree of temperature change.) The total deflections resulting from such small deformations will usually be too small to modify the shape of the structure materially.*

It may be further noted that it is seldom possible to determine the deflections of structures as they exist in practice to any great degree of refinement, nor is such refinement particularly desirable. It is a most important fact, and will be made clear in the later discussion, that in the analysis of indeterminate stresses it is the *relative* rather than the absolute values of the deflections which are important.

1*a*. *Methods of Analysis.*—The methods of determining deflections treated in this chapter may be classified as follows:

I. Method of Work.

a. The Maxwell-Mohr Method (Dummy Unit Loading).
b. Castigliano's Method (Derivatives of Internal Work).

* There are some important exceptions to the rule that the elastic deflections may be regarded as negligibly small in comparison to the main dimensions of the structure. For example, the calculated deflection of the Manhattan suspension bridge, under maximum live load, is about 15 ft.—roughly equal to 10 per cent of the sag and 1 per cent of the span. (See Johnson, Bryan and Turneaure's "Modern Framed Structures." Part II, page 247.)

II. Special Methods.
- *c*. The Moment-Area Method.
- *d*. The Elastic-Weight Method.
- *e*. The Displacement (Williot) Diagram.

Methoas *a* and *b* are based on the principle of the work of deformation, and we shall see later that they are nearly identical in mode of application to most problems here treated. We shall, therefore, group them under the head of the Method of Work, and we shall adopt this as the general basic method for the treatment of deflections. It is not always, and in fact not generally, the shortest or most direct method for dealing with special problems, but as a broad fundamental method for use in developing a comprehensive general theory, its advantages have led to a nearly universal adoption.

Methods *c* and *d*, despite a marked difference in fundamental conception, have so many points of similarity that they are frequently treated as a single method. They may be derived from the principle of work, but may also be established independently.

Method *e* is quite distinct from any of the others.

Still other means of finding deflections, differing markedly from any of the above and having wide fields of application, have been devised. Opinions as to advantages and disadvantages differ greatly. However, in an elementary treatise, we can only attempt to present some of the best known and most widely used methods of attack.

We have omitted from discussion the well-known and very important beam-deflection differential equation $\frac{d^2y}{dx^2} = \frac{M}{EI}$. It is assumed that the student is sufficiently acquainted with this method from his study of mechanics of materials.

SECTION I.—DEFLECTIONS BY METHOD OF WORK

2. Statement of Problem.—Before we proceed to a deduction of the deflection equation by this method, it is well to state the deflection problem in a somewhat different form, perhaps, from that with which the student is familiar, and to develop the conception of the work of deformation.

A beam AB deflects primarily because each elementary section dx is distorted, as shown in Fig. 8. (We shall later take up the relatively unimportant question of shearing deflections.)

We have, for our purpose, completely solved the problem of deflection for a straight beam when we answer the question, "If an element dx, distant x from A, has its faces distorted through the angle $d\alpha$, the

remaining portion of the beam assumed rigid, what is the corresponding displacement of any point q?" For, if the relationship can be established for *any* elementary section dx, the resulting displacement for all sections will be obtained by summing up the partial effects.

The corresponding problem in an articulated truss may be stated thus: "If any member S is deformed an amount ΔS, the remaining members assumed rigid, what is the corresponding displacement of a given point q?" (See Fig. 9.) If this relation is established for any

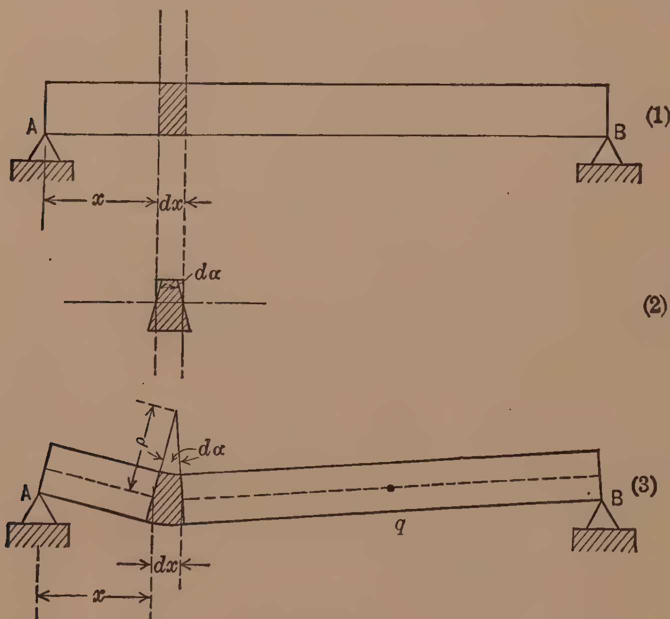


FIG. 8

member, the result for any number of members follows by direct summation.

We should further note that there is a *strictly geometrical* relation between the deformation of an element of a beam and the consequent displacement of any point, and between the change of length of a member of a truss and the resulting deflection of any point. That is to say, a given distortion will be connected with a certain displacement, *no matter what causes the distortion*. This fact is too obvious to need elaboration, but since the deduction here given of the general deflection equation is based in part upon it, the student should note it carefully. The principle of the work of deformation, to be developed in the next article, enables us to obtain conveniently a relation between internal dis-

tortion and the resultant displacement of any point, when the distortion is caused by a load at the point. But by the principle just stated, this relation must be true whatever be the cause of the distortion; hence we are able to generalize the result at once.

3. Internal Work of Deformation.—If a force is applied to any elastic body, there is a certain amount of energy expended in deforming the body. This must be equal to the product of the force times the component of the deflection of its point of application in the direction of its line of action, or to the sum of such products, if several loads are applied. If the elastic limit is not exceeded, the body will tend to regain its original position, and will do so against resistance, thereby perform-

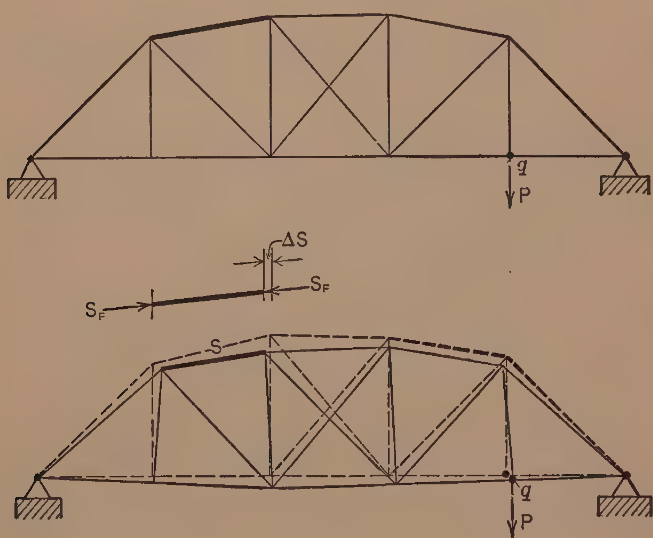


Fig. 9

ing a certain amount of work ("negative" work compared to that of the original deflecting forces). If the body is perfectly elastic,—we shall deal, here and later, only with strains inside the elastic limit,—it will recover fully its original shape and complete a cycle during which there must be no energy gained or lost; i.e., the strained body must give out as much energy in regaining its original state as was stored up in it during the process of deflection. This is a clear requirement of the law of conservation of energy.

The internal stored energy ("potential energy of strain") may be expressed mathematically as follows: Referring to Fig. 9, suppose a load P to be applied at q and all members except S assumed rigid. A

stress will be developed in each member of the truss which may be represented by a pair of external forces applied axially at the ends of the member. For all members except S, these forces maintain their relative positions during deflection, and hence do no work. The two forces S_F applied to the ends of the member S will obviously perform an amount of work equal to the product of the average value of S_F times the total deformation ΔS. If P be applied gradually, the stress S_F will gradually increase from zero to its full value, and the average value will be $\frac{1}{2} S_F$. The total internal work (only the one member S assumed deformable) will then be $\frac{1}{2} S_F \cdot \Delta S$. If L, A and E be given their usual significance, $\Delta S = \dfrac{S_F L}{AE}$ and the internal work of deformation may be expressed as

$$\Delta W_i = \tfrac{1}{2} S_F \cdot \Delta S = \frac{1}{2}\frac{S_F{}^2 L}{AE}. \qquad (1a)$$

If *all* members be regarded as elastic, this expression becomes

$$W_i = \tfrac{1}{2}\Sigma S_F \cdot \Delta S = \frac{1}{2}\sum \frac{S^2{}_F L}{AE}. \qquad (1)$$

A precisely analogous argument will hold for beams.

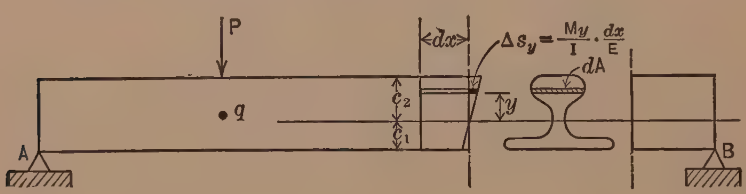


FIG. 10

Referring to Fig. 10, we suppose AB is a beam of any form of cross section subjected, let us say, to a transverse load P. The internal work of deformation for an element dx (remaining portion of beam assumed rigid) will evidently be the sum of the products of the various fiber deformations times the average value of the corresponding fiber stresses during deformation, if the load is gradually applied. Take the layer of fibers shown as the area dA in the figure; the deformation of each fiber is $\Delta s_y = \dfrac{My}{I} \cdot \dfrac{dx}{E}$ and if the fiber stress increases gradually from zero to its maximum, the average value will be $\dfrac{1}{2}\dfrac{My}{I} dA$ and the work of

deformation for this layer of fibers will therefore be $\frac{1}{2}\frac{M^2y^2dxdA}{EI^2}$ and the work of all the fibers on the cross section will be

$$\frac{1}{2}\int_{c_1}^{c_2}\frac{M^2dx}{EI^2}\cdot y^2dA = \frac{1}{2}\frac{M^2dx}{EI^2}\int_{c_1}^{c_2} y^2dA = \frac{1}{2}\frac{M^2dx}{EI}.$$

For the entire beam, the work will be obtained by summing the work performed by stresses at each element dx; hence

$$W_i = \frac{1}{2}\int_A^B\frac{M^2dx}{EI} = \frac{1}{2}\int_A^B\frac{Mdx}{EI}\cdot M = \frac{1}{2}\int_A^B d\alpha\cdot M, \quad . \; . \; (2)$$

if $d\alpha$ represents the angular change between the two faces of the sec-

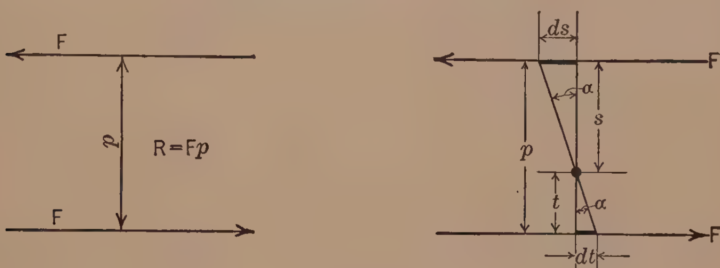


FIG. 11

tion dx. (The student will recall, from his study of mechanics of materials, that $\frac{d^2y}{dx^2} = \frac{M}{EI}$; therefore $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{EI} = \frac{d}{dx}\tan\alpha = \frac{d\alpha}{dx}$, since α is very small.)

We may show independently that if a couple R (see Fig. 11) is displaced in any manner, the work performed will be $R\alpha$ (if α is small). For evidently no work will be performed by pure translation of the couple, and the work of rotation may be expressed (if F maintains a constant value) as

$$F(ds) + F(dt) = Fs\alpha + Ft\alpha = F\alpha(s+t) = Fp\alpha = R\alpha. \quad . \; (3)$$

From the necessary relation of equality between internal and external work, we may say that if a beam is subjected to a number of loads P, so applied that the loads and corresponding internal stresses gradually increase from zero to the final value, and if δ in general represents the component deflection, in the direction of the load, of the point of application of any load P, then from Equation (2)

$$\tfrac{1}{2}\Sigma P\delta = W_i = \frac{1}{2}\int_0^L Md\alpha = \frac{1}{2}\int_0^L\frac{M^2dx}{EI}, \quad . \; . \; . \; . \; (4)$$

and similarly for trusses, if the members suffer axial stresses only, Equation (1) gives

$$\tfrac{1}{2}\Sigma P\delta = W_i = \tfrac{1}{2}\Sigma S\Delta s = \frac{1}{2}\sum\frac{S^2L}{AE}. \qquad (4a)^*$$

A. Deflections by Maxwell-Mohr Method (Dummy Unit Loading)

4. Truss Deflections.—We have seen that the essence of the problem of the deflection of structures is to obtain a relation between the distortion of a given element (small section dx of a beam or a single member of a truss) and the corresponding movement of a given point. The principle of work enables us to arrive at such a relation very simply. Applying the method to the truss of Fig. 9, load P applied gradually to point q and the member S alone regarded as deformable, Equation (4a) gives at once

$$\tfrac{1}{2}P\Delta\delta_q = \frac{1}{2}\frac{S^2L}{AE} = \frac{1}{2}S\cdot\Delta s,$$

whence

$$\Delta\delta_q = \Delta S\frac{S}{P}. \qquad (5a)$$

But we know from the fundamental theory of stresses that S is directly proportional to P, i.e.,

$$S = P \times \text{constant} = Pk, \text{ say.}$$

But this constant, k, is *numerically* equal to the value of S when P = unity. Following the usual notation, we shall call this stress S (due to unit load) $= u$. Then Equation (5a) becomes

$$\Delta\delta_q = \Delta S\cdot u; \qquad (5b)$$

or, if all members are deformable and S and ΔS are general terms for the stress and deformation of a member,

$$\delta_q = \Sigma\Delta S\cdot u. \qquad (5)$$

The relations (5) have been proved on the assumption that ΔS is caused by the load P at q. But from the discussion in the last paragraph of Article 2 it is clear that if $\delta_q = \Delta S\cdot u$ when ΔS is a deformation caused by a load at q, then δ_q must equal $\Delta S\cdot u$ when ΔS is the same change of length due to some other cause. Since Equation (5) holds for all finite values of P and A (so long as the deflections remain small), by suitable variation of these quantities we can make the equation

* These relations are due to Clapeyron, 1833.

cover the whole practical range of values of ΔS $\left(\Delta S = \frac{P}{A}\cdot\frac{uL}{E},\right.$ where u, L, and E are constants). Therefore, Equation (5b) is to be regarded as a perfectly general kinematical relation between any small change in length of a member and the corresponding displacement of a given point. ΔS may be a change of length due to temperature, to play in a pinhole, to the screwing up of a turnbuckle or a deformation due to a given loading. The latter is, of course, by far the most important case. If the truss is subjected to any set of loads and S designates the stress in any member due to these loads, we have

$$\delta_q = \Sigma \Delta S \cdot u = \sum \frac{SL}{AE}\cdot u = \sum \frac{SuL}{AE}, \quad . \quad . \quad . \quad . \quad (5c)$$

as usually written.

5. Beam Deflections.—The equation for beams follows similarly. We note that $\frac{1}{2}P\Delta\delta_q = \frac{1}{2}Md\alpha$ (see Fig. 10), if we consider only section dx elastic, and

$$\Delta\delta_q = \frac{M}{P}d\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

But $\frac{M}{P}$ is a constant and is numerically equal to the value of M when P is unity. Calling this value m, we have

$$\Delta\delta_q = m \cdot d\alpha, \quad . \quad . \quad . \quad (6b) \quad \text{or} \quad \delta_q = \int_0^L m d\alpha, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is the fundamental deflection equation for beams. We should note that in developing Equation (6a) we assume that the distortion of the faces of the element dx through the angle $d\alpha$ is produced by the load P. But we generalize the resulting relation as in the case of the truss; that is, a change $d\alpha$ at a given section will produce the same deflection at a point q regardless of what causes the change, and the deflection will be equal to the product of the angular change into the constant $\frac{M}{P} = m$.

In the case of the beam, we are concerned almost wholly with bending due to applied loads. For this case, $d\alpha = \frac{Mdx}{EI}$ and if the whole beam is treated as elastic (6) goes into

$$\delta_q = \int_0^L \frac{Mdx}{EI}\cdot m, \quad \text{or} \quad \delta_q = \int_0^L \frac{Mmdx}{EI}. \quad . \quad . \quad . \quad (6c)$$

6. Deflection Constants.—Equations (5) and (5c), (6) and (6c) give a general solution of the problem of deflections. The quantities u and m may be termed the *deflection constants* for trusses and beams respectively. When put into words, the deflection equation for trusses states that if a bar changes its length by an amount ΔS, the corresponding displacement of any joint q in any desired direction is equal to ΔS times the deflection constant u, the latter being numerically equal to the stress in the member due to a unit load at q, acting in the direction of the displacement desired.

For beams, we may say that if any element of the beam undergoes a relative angular displacement of its faces, $d\alpha$, the corresponding displacement in any given direction of any point q in the axis of the beam is $d\alpha$ times the deflection constant m, which is here equal to the moment at the section where the element is taken produced by a unit load at q acting in the direction of the displacement sought.

7. General Interpretation of δ.—*Angular Displacement.* In the foregoing discussion, δ has been used to signify the linear displacement of a point referred to its original position. But there are other cases of deflection which are of considerable importance. For instance, we may wish to know the relative displacement of two points with respect to each other, or we may wish to know the angular displacement of a given line in a beam or framework. The given equations may be at once generalized to cover these cases by a proper interpretation of the deflection constant.

Referring to Fig. 12a, let us suppose that a pair of loads P, P act at b and C as shown, and that member BC (any other member might have been selected) alone is elastic. From Equation (4a), if $\Delta\delta$ = relative displacement of b and C along line bC,

$$\tfrac{1}{2}P\Delta\delta = \tfrac{1}{2}S\Delta S, \quad \text{or} \quad \Delta\delta = \frac{S}{P}\Delta S = u\cdot\Delta S, \quad . \quad . \quad . \quad (7a)$$

which is identical with (5b) if by u we understand the stress in BC due to a *pair* of unit forces acting as shown in the figure.

Similarly, suppose a couple $F\cdot L = M$ to be applied to the line Bc as shown in Fig. 12a, and again imagine all members rigid except BC, and let S and ΔS be the stress and deformation, respectively, of BC due to the couple M. Then, in view of the equality of internal and external work and of Equation (3), (if $\Delta\alpha$ is the small angle through which bar Bc is displaced),

$$M\Delta\alpha = S\Delta S, \quad \text{or} \quad \Delta\alpha = \frac{S}{M}\Delta S = u\cdot\Delta S. \quad . \quad . \quad . \quad (7b)$$

which is analogous to (5b) if u is the stress in a given member *due to a*

couple of magnitude unity, applied to a line whose angular displacement $\Delta\alpha$ is desired.

In Fig. 12b, suppose a couple M_q is applied to the beam at q, and only the element dx is elastic. Let $\Delta\alpha_q$ be the angular change of the line $1-1$ caused by $d\alpha$, the angular change between the faces of the element dx, due to the moment M at section $2-2$. (This moment is produced by the applied moment M_q.) Then

$$M_q\Delta\alpha_q = M d\alpha, \quad \text{and} \quad \Delta\alpha_q = \frac{M}{M_q} d\alpha = m d\alpha, \quad . \quad . \quad . \quad (8)$$

if m is the moment at section $2-2$ *due to a couple of magnitude unity at* $1-1$. Since, as already noted, the only flexural displacement of

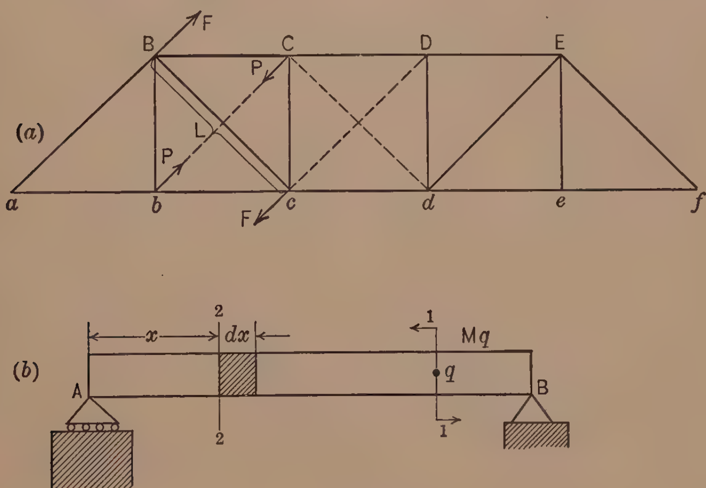


FIG. 12

any importance in beams is that due to stress, we may give $d\alpha$ its value, $\frac{Mdx}{EI}$, and Equation (8) will read

$$\Delta\alpha_q = \frac{Mmdx}{EI}, \quad \text{or} \quad \alpha_q = \int_A^B \frac{Mmdx}{EI},$$

if all sections are elastic. This is analogous to (6c), the only change being in the character of m.

These equations are derived on the assumption that the applied loading is the cause of the internal deformation. But we may show, exactly as in the preceding cases, that the relation holds whatever the cause of ΔS and $d\alpha$.

We may then make the following general statement regarding the deflection equations:

(1) Beams. If any element dx, in a beam AB, is deformed after the manner of Fig. 8, a displacement will in general take place at each point or section of the beam; this displacement will be in direct proportion to the deformation, and its amount will be equal to the relative angular displacement of the two faces of the element dx multiplied by a constant "deflection factor." This constant is numerically equal to the moment at dx produced by a *unit loading* applied at the point or section whose displacement is under consideration. This unit loading will be a single force equal to unity if linear deflection is sought, and must act in the direction of the deflection. It will be a couple of magnitude unity if angular deflection is desired.

(2) Trusses. If any member of a jointed frame changes its length a small amount, each point of the frame will in general be displaced, and the amount of the displacement will be equal to the change of length multiplied by the deflection factor for the point and member. This deflection constant is numerically equal to the stress in the member caused by a *unit loading* applied at the point or section where the displacement is desired. As for beams, the unit loading will be a unit force or unit couple, depending on whether linear or angular displacement is desired. To emphasize the distinction, we shall generally use the symbol δ to represent linear deflection and α to represent angular deflection, but it is clear that we might very well use δ (or some other symbol) as a *perfectly general* designation of elastic displacement—linear or angular, depending upon the nature of the deflection constants.

7a. *Units.*—It is well to keep in mind the units involved in the various terms of the deflection equation. Thus

$$\delta \times 1\# = \int \frac{Mmdx}{EI}, \quad \text{or} \quad \delta = \int \frac{Mmdx}{EI \times 1\#},$$

gives

$$\text{ins.} = \frac{(\text{lbs.} \times \text{ins.})\ (\text{lbs.} \times \text{ins.}) \times \text{ins.}}{\frac{\text{lbs.}}{\text{ins.}^2} \times \text{ins.}^4 \times \text{lbs.}} = \text{ins.}$$

Similarly

$$\delta \times 1\# = \sum \frac{SuL}{AE}, \quad \text{or} \quad \delta = \sum \frac{SuL}{AE \times 1\#},$$

gives

$$\text{ins.} = \frac{\text{lbs.} \times \text{lbs.} \times \text{ins.}}{\text{ins.}^2 \times \frac{\text{lbs.}}{\text{ins.}^2} \times \text{lbs.}} = \text{ins.}$$

For convenience in writing the equation, we ordinarily omit the term representing the unit load; i.e., we say δ is *numerically* equal to

$$\sum \frac{SuL}{AE}, \text{ or equal to } \int \frac{Mmdx}{EI}.$$

Or we may think of the deflection constants u and m as respectively equal to

$$\frac{\left\{\begin{matrix}\text{Stress due to unit loading applied at deflected point}\\ \text{in direction of desired deflection}\end{matrix}\right\}}{\text{Unit loading}}$$

and

$$\frac{\left\{\begin{matrix}\text{Moment due to unit loading applied at deflected point}\\ \text{in direction of desired deflection}\end{matrix}\right\}}{\text{Unit loading}}$$

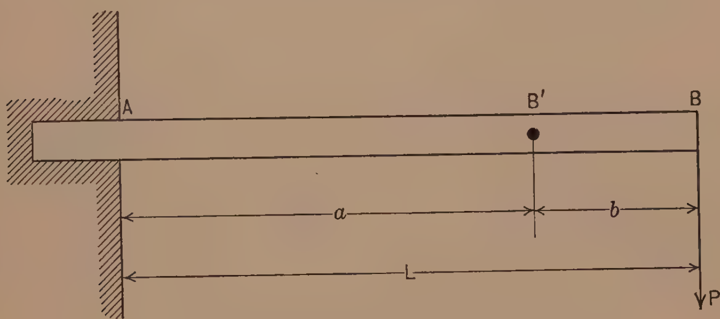


FIG. 13*a*

8. Examples.

(1) *Deflection of cantilever loaded at end* (Fig. 13*a*).

$$\delta_B{}' = \int_A^B \frac{Mmdx}{EI};$$

with origin at B,

$$\left.\begin{aligned} M &= Px \\ m &= o,\ x < b \\ m &= 1 \times (x - b),\ x > b \end{aligned}\right\} Mmdx = (Px^2 - Pbx)dx$$

$$\delta_B{}' = \frac{P}{EI}\int_b^L (x^2 - bx)dx = \frac{P}{EI}\left[\frac{x^3}{3} - \frac{bx^2}{2}\right]_b^L = \frac{P}{3EI}\left[L^3 - \frac{3L^2b}{2} + \frac{b^3}{2}\right].$$

$$\text{If } b = o,\ \delta_B = \frac{PL^3}{3EI},$$

$$\alpha_{B'} = \int_B^A \frac{Mm_\alpha dx}{EI} = \frac{P}{EI}\int_b^L xdx = \frac{P}{2EI}(L^2 - b^2),\ (\text{since } m_\alpha = \text{unity}).$$

$$\text{If } b = o,\ \alpha_B = \frac{PL^2}{2EI}.$$

(2) *Cantilever uniformly loaded* (Fig. 13*b*).

$$\delta_{B'} = \int_A^B \frac{Mmdx}{EI},$$

$$M = \frac{wx^2}{2}; \quad m = \begin{cases} o, x < b \\ x - b, x > b \end{cases}$$

$$Mm = \frac{wx^3 - wbx^2}{2} dx,$$

$$\delta_{B'} = \frac{w}{2EI}\int_b^L (x^3 - bx^2)dx = \frac{w}{24EI}[3L^4 - 4bL^3 + b^4].$$

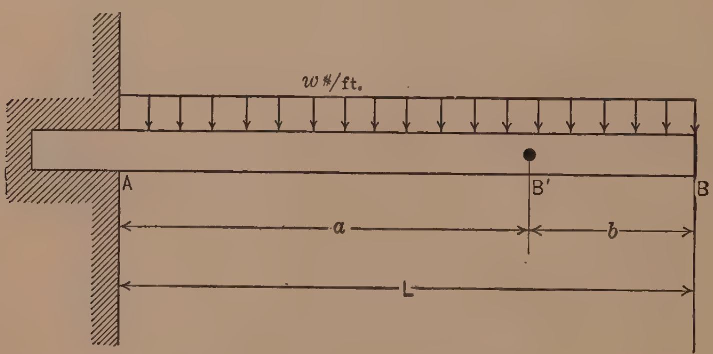


Fig. 13*b*

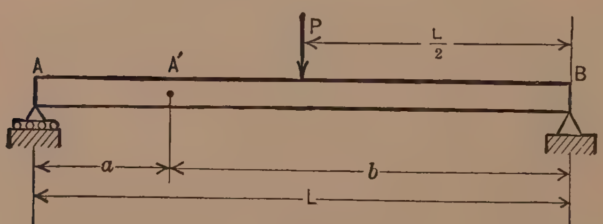


Fig. 13*c*

If $b = o$, $\qquad \delta_B = \dfrac{wL^4}{8EI}$,

$$\alpha_B = \int_A^B \frac{Mm_\alpha dx}{EI} = \frac{w}{2EI}\int_b^L x^2dx = \frac{w}{6EI}(L^3 - b^3).$$

If $b = o$, $\qquad \alpha_B = \dfrac{wL^3}{6EI}$.

(3) *Simple beam with single load at center* (Fig. 13c).

$$\delta_{A'} = \int_A^B \frac{Mmdx}{EI}.$$

Origin at B, $M = \frac{Px}{2}$, if $x < \frac{L}{2}$,

$$m = \frac{a}{L}x, \text{ if } x < \frac{L}{2}.$$

Origin at A, $M = \frac{P}{2}x$, if $x < \frac{L}{2}$,

$$m = \begin{cases} \frac{b}{L}x, \text{ if } x < a, \\ \frac{b}{L}x - (x - a) = \frac{a(L - x)}{L}, \text{ if } \frac{L}{2} > x > a. \end{cases}$$

$$\therefore \quad \delta_{A'} = \frac{Pa}{2EIL}\int_0^{\frac{L}{2}} x^2dx + \frac{Pb}{2EIL}\int_0^a x^2dx + \frac{Pa}{2EIL}\int_a^{\frac{L}{2}} x(L - x)dx$$

$$= \frac{Pa}{48EI}(3L^2 - 4a^2).$$

If $a = \frac{L}{2}$, $\delta_{\max} = \frac{PL^3}{48EI}$.

9. Maxwell's Law of Reciprocal Deflections.—In the beam of Fig. 14*a* let *any* two points p and q carry equal loads P. If we suppose the load at q to be removed and write the equation for the deflection at q due to P at p we have,

$$\delta_{qp} = \int_0^L \frac{M_p m_q dx}{EI},$$

where M_p = moment at any section due to P acting at p;
and m_q = moment at any section due to a unit load acting at q.

Similarly, if the load at p is removed and we have P acting at q alone

$$\delta_{pq} = \text{deflection at } p \text{ due to } P \text{ acting at } q = \int_0^L \frac{M_q m_p dx}{EI}.$$

But

$$M_p = Pm_p \quad \text{and} \quad M_q = Pm_q,$$

whence

$$\delta_{pq} = \delta_{qp}. \qquad (9)$$

that is, given any two points in a beam, the deflection at the first due to a given load acting at the second, is equal to the deflection at the second due to the same load acting at the first.

A similar argument establishes the theorem for a truss (Fig. 14*b*).

This principle, known as "Maxwell's Law of Reciprocal Deflections," * is one of the most useful and important in the theory of indeterminate structures. It is more general than appears from the preceding illustration. It obviously applies, as the student may easily show, to loads having different directions. It applies also to angular as well as to linear displacements, as may easily be shown. We may first note that since the magnitude of the equal loads P is immaterial, it will be convenient to take $P = 1$ #, and the theorem is then con-

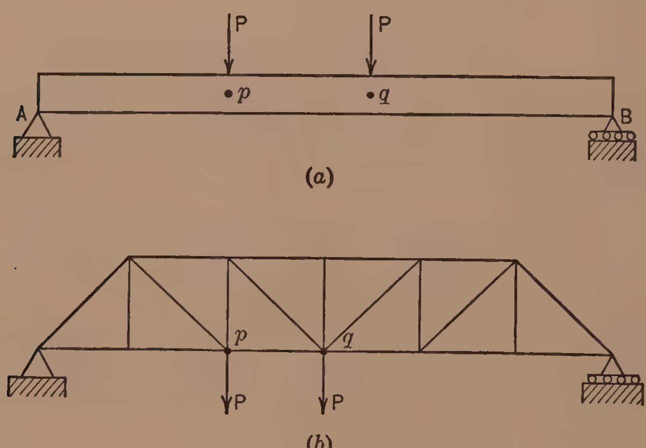


Fig. 14

veniently stated thus—"the deflection at p due to unity at q = deflection at q due to unity at p." If in place of "unit load" we put unit couple, we at once obtain

$$\alpha_{qp} = \int_0^L \frac{m\dagger_{(\alpha)q} \cdot m_{(\alpha)p}\,dx}{EI} = \int_0^L \frac{m_{(\alpha)p} \cdot m_{(\alpha)q}\,dx}{EI} = \alpha_{pq} \quad . \quad (10)$$

Further, if we suppose a single load unity acting at p in any direction

* After its discoverer, James Clerk Maxwell, Cavendish Professor of Experimental Physics at Cambridge University, and one of the greatest physicists of modern times.

† The symbol "$m_{(\alpha)q}$" is used here to emphasize the fact that m is due to a unit couple rather than a single unit force. The student should note that if we define m_q as the moment at any given section due to a unit *loading* at q, this covers both the above cases and no special symbol is needed—see remarks on page 22, (2).

and we wish the *angular* displacement at q, by application of the fundamental formula we get

$$\alpha'_q \text{ (due to unit } load \text{ at } p) = \int_0^L \frac{m_p \cdot m_{(\alpha)q} dx}{EI},$$

where m_p = moment at any section due to unit load at p, as before;

and $m_{(\alpha)q}$ = moment at any section due to unit couple at q.

If we have a couple unity acting at q, no other loads, and we wish the linear displacement at p (in the direction of above unit load) we must have

$$\delta'_p \text{ (due to unit } couple \text{ at } q) = \int_0^L \frac{m_{(\alpha)q} m_p dx}{EI} = \alpha'_q, \quad . \quad (11)$$

that is, "the angular displacement at q due to a unit load acting in a given direction at p, is equal to the linear displacement (in this direction) at p due to unit couple at q." This holds equally for a truss.

10. Shearing Deflection.—In the presentation of the theory of deflections in the preceding pages, no mention has been made of deformation due to shear. We may investigate this problem in a manner similar to that used for the bending deflections. Referring to Fig. 15*a*, we proceed to find the deflection at q due to a load P at q, assuming only the section dx as elastic. From the equality of internal and external work,

$$P \times \Delta\delta_q = \text{Internal work due to shear at section } dx$$
$$= V_P \cdot f, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

if we assume that the shearing stress is uniformly distributed over the cross-section. This equation is analogous to Equation (6*b*). We may show that $\frac{V_P}{P}$ is a constant, numerically equal to the shear at the section when P = unity. Calling this deflection factor for shear v (corresponding to m for bending and u for axial stress in a truss member), we get

$$\Delta\delta_q \text{ (due to shearing distortion in } dx) = v \times f, \quad . \quad . \quad . \quad (12a)$$

where v is the shear at section dx due to a unit load at q acting in the direction of the desired δ_q. The formula is general and will give the displacement of the point q due to a shear in dx *from any cause.*

From the study of strength of materials we know that if v_s = unit shearing stress, G = shearing modulus of elasticity and γ = unit de-

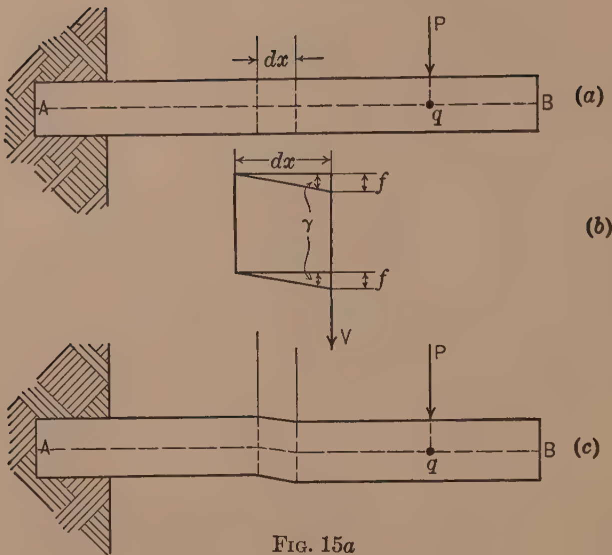


FIG. 15*a*

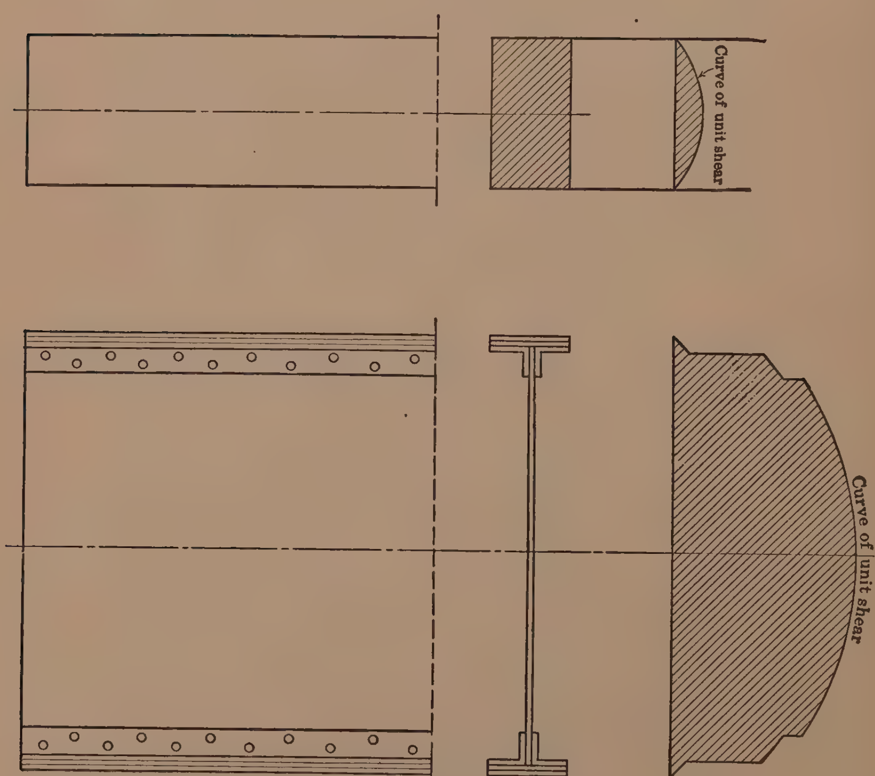


FIG. 15*b*

trusion, or angle of shear, we must have $\gamma = \frac{v_s}{G} = \frac{V}{AG}$, if V is the total shear at section due to given loading. Also since f is small, we have

$$f = dx \cdot \gamma = \frac{Vdx}{AG}.$$

For the shear throughout the length of the beam,

$$\delta_q = \int_0^L \frac{Vvdx}{AG}. \qquad (13)$$

Expressions for angular displacement due to shear may be deduced in a similar manner, but this is of little practical significance.

For a simple beam loaded with P at the center, the center deflection is

$$\delta_c = 2\int_0^{\frac{L}{2}} \frac{Vvdx}{AG} = 2\int_0^{\frac{L}{2}} \frac{\frac{P}{2}\cdot\frac{1}{2}dx}{AG} = \frac{PL}{4AG}.$$

For uniform load w per unit of length,

$$\delta_c = 2\int_0^{\frac{L}{2}} \frac{Vvdx}{AG} = 2\int_0^{\frac{L}{2}} \frac{\left(\frac{wL}{2} - wx\right)\cdot\frac{1}{2}dx}{AG} = \frac{wL^2}{8AG}.$$

Comparing these results with the corresponding center deflections due to flexure, we note that

$$\frac{\frac{PL}{4AG}}{\frac{PL^3}{48EI}} = \frac{\frac{PL}{16AE}}{\frac{PL^3}{48EAr^2}} = 30\frac{r^2}{L^2},$$

if $G = \cdot 4E$ which is approximately correct for steel; also

$$\frac{\frac{wL^2}{8AG}}{\frac{5wL^4}{384EI}} = 24\left(\frac{r}{L}\right)^2,$$

with similar assumptions.

For I beams and plate girders, r is approximately $\frac{1}{2}d$. For rectangular sections, r equals $\frac{d}{\sqrt{12}}$.

The following tabulation shows the relative importance of shear and moment deflections for different ratios of $\frac{d}{L}$.

TABLE A

SHEARING DEFLECTION TO MOMENT DEFLECTION—PER CENT

$\frac{d}{L}$	I-type of Section		Rectangular Section	
	Uniform Load	Concentrated Load at Center	Uniform Load	Concentrated Load at Center
$\frac{1}{5}$	30	24	10	8
$\frac{1}{10}$	7.5	6	2.5	2.0
$\frac{1}{15}$	3.33	2.66	1.11	0.9

A majority of beam and girder spans have a proportionate depth of less than $\frac{1}{10}$, and for such cases the tabulation shows that no serious error will be involved in neglecting the shearing deflection. For short, deep beams and for trusses (where the proportionate depth is $\frac{1}{5}$ to $\frac{1}{7}$) the shear deflection cannot safely be ignored. In all cases of girders and beams with solid webs treated in this book, the deflection due to shear will be neglected.

We should note again that these comparisons are made on the assumption that the shear distribution across the section is uniform. The actual distribution for rectangular and I sections is shown in Fig. 15*b*. This results in a greater proportional deflection of the neutral plane due to shear, especially for the I section, but it does not invalidate the general conclusion stated above.

11. General Equations for Combined Axial, Flexural and Shearing Stresses.—If we have a bar subjected to both transverse and longitudinal loads, we may express the total resultant displacement of an arbitrary point (see Fig. 16) in any specified direction by the superposition of the separate effects due to thrust, bending, and shear. We have

$$\delta_q = \frac{SuL}{AE} + \int_0^L \frac{Mmdx}{EI} + \int_0^L \frac{Vvdx}{AG}. \quad . \quad . \quad . \quad (14)$$

$$\alpha_q = \frac{Su_\alpha L}{AE} + \int_0^L \frac{Mm_\alpha dx}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14a)$$

These formulas assume that S and A are constants, which is usually the case. If either or both are variable, we must write

$$\int_0^L \frac{Sudx}{AE} \text{ instead of } \frac{SuL}{AE}.$$

This expression is often written in a different notation. If $S = N$ (normal force) and $u = n$ (axial stress due to unit loading at point of deflection) we have

$$\int_0^L \frac{Sudx}{AE} = \int_0^L \frac{Nndx}{AE},$$

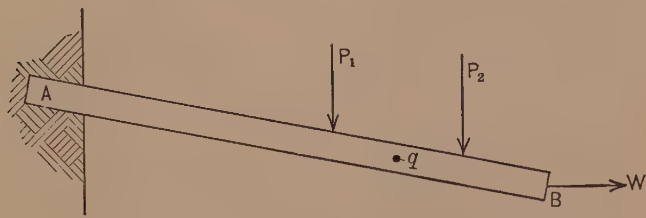


Fig. 16

which is the form commonly adopted in dealing with slightly curved bars, and which will be used later in this book.

We frequently meet with a type of framework in which some of the members are subjected to bending as well as axial stress. In Fig. 17, members FE, FC, DC are hinged at their ends and hence receive axial stress only. But members ADF and BCE are continuous from A to F and B to E, and in general will be subjected to both direct stress and bending.

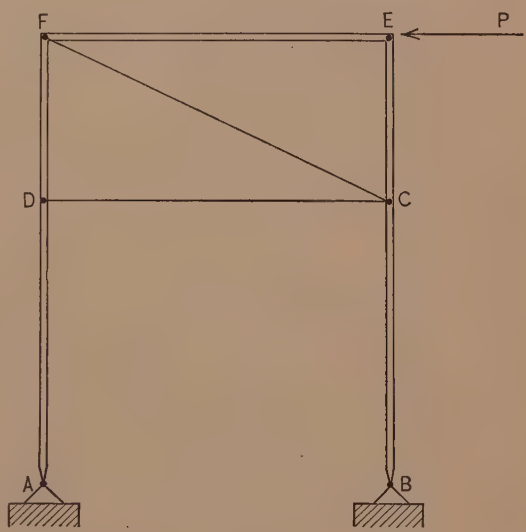


Fig. 17

For such a case we obviously have for the deflection equation,

$$\delta = \sum \frac{SuL}{AE} + \sum \int^L \frac{Mmdx}{EI}, \quad . \quad . \quad . \quad . \quad (15)$$

$$\alpha = \sum \frac{Su_\alpha L}{AE} + \sum \int_0^L \frac{Mm_\alpha dx}{EI} \quad . \quad . \quad . \quad . \quad (15a)$$

The terms involving $\sum \int_0^L$ mean that for a member subjected to

bending, we integrate the expression $\frac{Mmdx}{EI}$ from one end of the member to the other, and if there are several such members, add the results.

12. Deflection of Curved Bars.—Thus far we have dealt with the deflection of straight beams, and frames composed of straight bars. Many important cases arise in the theory of structures (the arch rib, for example) in which formulas expressing the distortion of curved bars are required. This problem falls under two cases: (1) the case where the radius of curvature of the axis of the bar, and the depth of the bar in the plane of bending are quantities of the same order of magnitude; and (2) the case where the curvature is slight and the radius of curvature may be considered a very large number compared to the depth of the cross section.

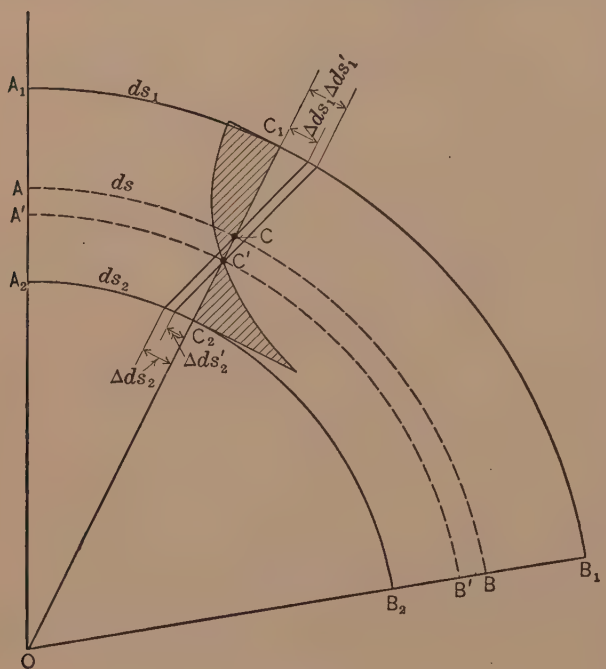


Fig. 18

In the first case we cannot assume that the simple stress distribution of the straight bar is even approximately true. In the curved beam of Fig. 18, which we will assume to have a symmetrical section, the length of the lower fibers ds_2 is much less than that of the upper fibers ds_1, and, assuming the distorted cross-section to remain plane and the neutral axis to be in the mid-plane, $\frac{\Delta ds_2}{ds_2}$, which measures the bottom fiber stress, must be greater than $\frac{\Delta ds_1}{ds_1}$ which measures the top fiber stress ($\Delta ds_1 = \Delta ds_2$); whence it is clear that we cannot maintain equilibrium with the neutral axis in the central plane. As a matter of fact, the axis shifts toward the lower side, and the stress distribution takes the form shown in the

hatched area—a hyperbolic curve. For hooks, links, thick rings, and similar problems, the analysis must be carried out on this basis. On the other hand, for the case of the arch rib or most other curved bars met with in structural design, where the radius of curvature is from 15 to 30 times the depth, the relative variation between the upper and lower fiber length is slight and the stress distribution is sensibly linear, so that for symmetrical sections the neutral axis may without serious error be taken to coincide with the neutral plane (Fig. 19).

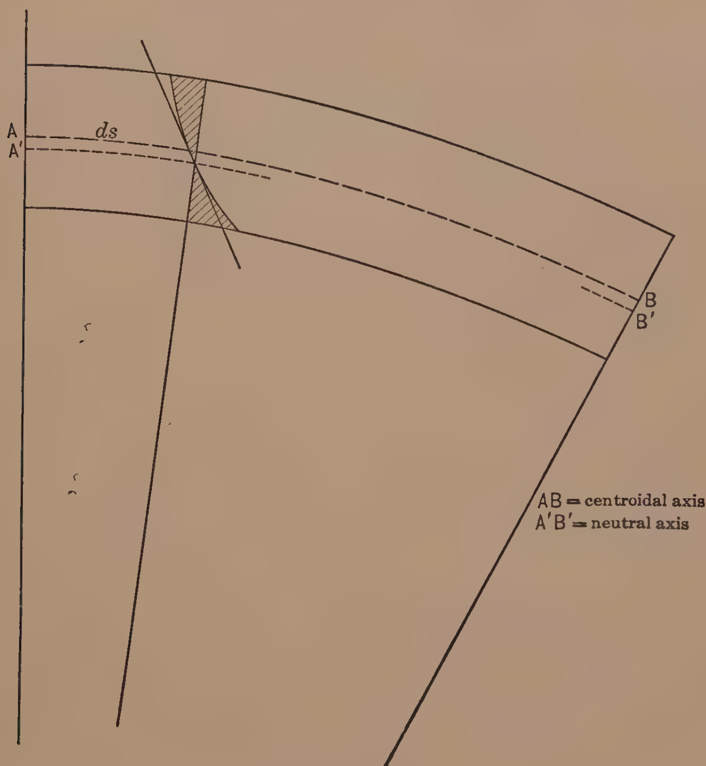


Fig. 19

In the present treatise we shall deal with curved bars of the latter type only. The deflection equations are easily obtained by a method similar to that used for straight bars.

Let AB (Fig. 20) be the section of a curved bar acted upon by loads (not shown in figure) which induce both axial stress and bending. Assuming for the moment that only a small section of length ds is elastic, we wish to find the displacement of the point B due to this distortion. This latter will in general consist of a shortening or lengthen-

ing of ds by an amount Δds and a rotation of the face C_1C_2 through the small angle $\Delta d\phi$. As in the corresponding case for the straight beam, we assume a unit load acting at B in the direction of the desired deflection (vertical in figure). This will in general produce a moment m and and an axial stress, n at every section. Since we are temporarily regarding

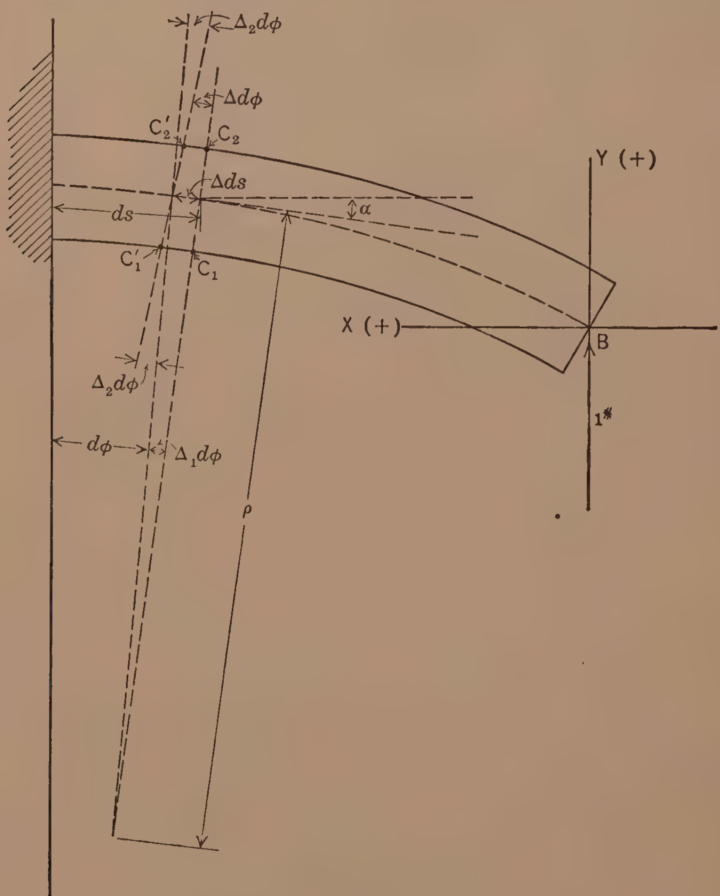


FIG. 20

all portions of the beam as rigid except the length ds, the internal work *due to the* 1 # *load* when the small portion ds is deformed as above, *from any cause*, will be (from the fundamental formulas)

$$\Delta W_i = n \cdot \Delta ds + m \cdot \Delta d\phi = 1\,\# \cdot \Delta \delta_B,$$

since the internal work of the unit load must equal the corresponding external work.

If all sections are elastic

$$1 \maltese \cdot \delta_B = \int_A^B n \cdot \Delta ds + \int_A^B m \cdot \Delta d\phi.$$

This formula is perfectly general, but ordinarily we deal with the case where Δds and $\Delta d\phi$ are deformations due to a specified loading. In such a case, if we call the resultant moment at any section M and the resultant normal stress through the axis N, we shall have

$$\Delta ds = \frac{Nds}{AE}; \ (a)$$

and $\Delta d\phi$ = angular change due to axial deformation + angular change due to bending

$$= \Delta_1 d\phi + \Delta_2 d\phi = \frac{Nds}{AE\rho} + \frac{Mds}{EI}; \ (b)$$

$\left(\text{since from Fig. 20, } \rho\Delta_1 d\phi = \Delta ds, \text{ whence } \Delta_1 d\phi = \frac{Nds}{AE\rho} \text{ from (a)}\right).$

It will be noted that these expressions are identical with the equations for a straight bar except for the added term $\frac{Nds}{AE\rho}$. This addition arises from the fact (which will be clear from the figure) that where the axis of the bar is curved, a displacement of the face C_1C_2 by an amount Δds along the axis must always be accompanied by a corresponding angular change $\Delta d\phi = \frac{\Delta ds}{\rho}$ even if there is no bending. We may write the formula finally (dividing out the $1 \maltese$):

$$\delta_B = \int_A^B \frac{Nnds}{AE} + \int_A^B \frac{Nmds}{AE\rho} + \int_A^B \frac{Mmds}{EI}, \quad . \quad . \quad . \quad (16)$$

or, if we let $\maltese = n + \frac{m}{\rho}$.

$$\delta_B = \int_A^B \frac{N\maltese ds}{AE} + \int_A^B \frac{Mmds}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16a)$$

If the section A is constant from B to A, and ρ approaches infinity,

$$ds = dx; \int_A^B \frac{Nnds}{AE} = \frac{NnL}{AE}\left(= \frac{SuL}{AE}\right); \int_A^B \frac{Nmds}{AE\rho} = 0,$$

and (16) becomes

$$\delta_B = \frac{SuL}{AE} + \int_0^L \frac{Mmdx}{EI},$$

which is Equation (14) with the shearing effect omitted.

Again referring to Fig. 20, if we choose the positive directions of coordinate axes as there shown, and designate counterclockwise rotation as positive (moments will then be positive which compress the top fibers), and if we further put $\frac{N}{A} = s$ and recall that $dx = ds \cdot \cos\alpha$ and $dy = ds \cdot \sin\alpha$ where α is the inclination of the tangent to the axis of the beam to the axis of x, we may write

$$\text{Vertical deflection} \quad = \delta_y = \int_A^B \left[\frac{sdy}{E} + \frac{sxds}{E\rho} + \frac{Mxds}{EI}\right]. \quad . \quad . \quad (17a)$$

$$\text{Horizontal deflection} = \delta_x = \int_A^B \left[\frac{sdx}{E} - \frac{syds}{E\rho} - \frac{Myds}{EI}\right]. \quad . \quad . \quad (17b)$$

These follow from (16), substituting for case of vertical deflection

$$m = 1\# \cdot x, \quad n = 1\# \cdot \sin\alpha,$$

and for horizontal deflection,

$$m = -1\# \cdot y, \quad n = 1\# \cdot \cos\alpha.$$

The signs will appear correct from physical considerations if it be noted that any positive rotation $\Delta d\alpha$ displaces B *upward* and *outward.*

B. Deflection as the Partial Derivative of the Internal Work of Deformation

13. General Equations.—Let AB, Fig. 21, be any beam or truss acted upon by any group of loads.

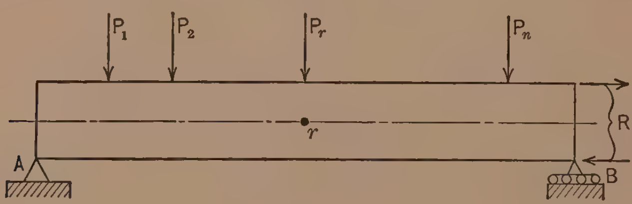


Fig. 21

We have (if loads are gradually applied, increasing uniformly from zero to P) W_i = internal work of deformation = external work of applied loads

$$= \tfrac{1}{2}P_1\delta_1 + \tfrac{1}{2}P_2\delta_2 + \ldots \tfrac{1}{2}P_n\delta_n. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

We inquire what is the change in W_i if any load, as P_r changes by a very small amount ΔP_r. Since we assume that the elastic effect of

each load is independent of the others, it is clearly a matter of indifference how the increment to P_r is applied. We may (1) apply the loads $P_1, P_2 \ldots (P_r + \Delta P_r) \ldots P_n$ simultaneously; (2) we may apply the loads P and later add to them the small load ΔP_r; or (3) we may apply ΔP_r first and then apply the loads P. The final result must be the same in each case. Therefore, assuming the latter order and gradual application of loads, we shall have

$$W_i + \Delta W_i = \tfrac{1}{2}\Delta P_r \cdot \Delta \delta_r + \tfrac{1}{2}\Sigma P\delta + \Delta P_r \cdot \delta_r.$$

If we take ΔP_r sufficiently small, $\Delta P_r \cdot \Delta \delta_r$ vanishes to the second order of small magnitudes, and recalling (18) we have

$$\Delta W_i = \Delta P_r \cdot \delta_r, \quad \text{or} \quad \delta_r = \frac{\Delta W_i}{\Delta P_r} = (\text{in the limit}) \frac{\partial W_i}{\partial P_r}. \quad . \quad (19)^*$$

That is to say, in any beam or truss subjected to any set of loads, the deflection of an arbitrary point r is equal to the first partial derivative of the internal work of deformation with respect to a load at the point, P_r, which acts in the direction of the desired deflection.

It should be noted that the right-hand member of (19) expresses a (partial) rate of change of the internal work as the load P_r changes. It is perfectly general for all finite values of the loads, and *includes the case where a load is zero.* We write down the general algebraic expression for the total internal work and form its first partial derivative with respect to *an arbitrary load* acting at the specified point. In this expression for the derivative we substitute the actual value of the load acting at r. If, as is frequently the case, r is a point at which there is no load, or none having a component in the direction of the desired deflection, P_r is equated to zero. A very simple example will serve to clear up the method.

14. Application to Problem of Linear Displacement.—Let it be required to find the vertical deflection at B due to a load P at B, Fig. 22. From "Mechanics of Materials" (see Eq. (2), p. 17)

$$W_i = \frac{1}{2}\int_0^L \frac{M^2dx}{EI} = \frac{P^2}{2EI}\int_0^L x^2dx = \frac{P^2L^3}{6EI}$$

(assuming I constant), and $\dfrac{\partial W_i}{\partial P} = \dfrac{PL^3}{3EI}$, the well-known expression for the maximum deflection of a cantilever with a single load at the end.

If we wish the vertical deflection at some intermediate point, as r,

* This derivation follows closely that given by Föppl, "Vorlesungen," III, pp. 167–69.

we imagine an additional vertical load P_r applied to the beam. Then the total work is

$$W_i = \frac{1}{2}\int_0^{x_1} \frac{(Px)^2dx}{EI} + \frac{1}{2}\int_{x_1}^{L} \frac{[(P+P_r)x - P_rx_1]^2dx}{EI}.$$

The first term, being independent of P_r, will disappear on differentiation, and hence may for our purpose be omitted.

$$\frac{1}{2EI}\int_{x_1}^{L}[(P+P_rx) - P_rx_1]^2dx$$

$$= \frac{1}{2EI}\left[(P+P_r)\frac{L^3 - x_1^3}{3} - PP_rx_1(L^2 - x_1^2) + P_r^2x_1^2(L - x_1)\right].$$

$$\therefore \frac{\partial W_i}{\partial P_r} = \frac{1}{2EI}[\tfrac{2}{3}(P+P_r)(L^3 - x_1^3) - Px_1(L^2 - x_1^2) + P_rx_1^2(L - x_1)].$$

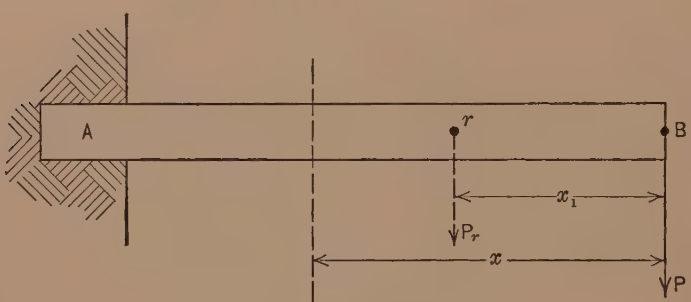


FIG. 22

This is a general formula, valid for any values (not infinite) of P and P_r. In this case $P_r = 0$ and

$$\frac{\partial W_i}{\partial P_r} = \delta_r = \frac{P}{6EI}[2L^3 - 3L^2x_1 + x_1^3], \ldots \text{ a well-known result.}$$

15. Angular Displacement.—We may without difficulty extend the above method to the case of angular displacement. If in the structure of Fig. 21 we have a couple, R, acting at the end section B, for example, the work equation becomes,

$$W_i = \tfrac{1}{2}\Sigma P\delta + \tfrac{1}{2}R\alpha,$$

if α is the angular displacement at B. If now we imagine an increment ΔR to be added to the above loading, and applied *before* the other loads, just as in the preceding case,

$$W_i + \Delta W_i = \tfrac{1}{2}\Sigma P\delta + \tfrac{1}{2}R\alpha + \Delta R\cdot\alpha,$$

whence

$$\alpha = \frac{\Delta W_i}{\Delta R} = \text{(in the limit) } \frac{\partial W_i}{\partial R}, \quad . \quad . \quad . \quad (20)$$

that is to say, the angular displacement at any section of a girder is equal to the first partial derivative of the internal work with respect to a couple R acting at the section. As in the case of linear displacement, it is unnecessary that the actual applied loading shall include such a couple; we obtain $\frac{\partial W_i}{\partial R}$ in a manner analogous to the preceding case. If in the beam of Fig. 22 we have a couple R acting at B

$$W_i = \frac{1}{2}\int_0^L \frac{M^2dx}{EI} = \frac{1}{2EI}\int_0^L (Px + R)^2dx$$

$$= \frac{1}{2EI}\left[\frac{Px^3}{3} + PRx^2 + R^2x\right]_0^L$$

$$= \frac{1}{2EI}\left[\frac{PL^3}{3} + PRL^2 + R^2L\right].$$

$$\frac{\partial W_i}{\partial R} = \frac{1}{2EI}\left[PL^2 + 2RL\right],$$

which, for $R = 0$, gives

$$\frac{\partial W}{\partial R} = \alpha = \frac{PL^2}{2EI},$$

another well-known result.

16. Summary and Comparison.—The above principle is one of great generality and importance in its application to the theory of structures and it is usually referred to as "Castigliano's first theorem." *

We should note the important limitation that, as above expressed, the theorem can be directly applied only to structures with rigid supports or at least where the reactions perform no work.†

To compare the expressions (19) and (20), radically different in form from the previous deflection equations, with these latter, we observe that since the internal work in a bar due to axial stress and flexure resulting from the gradual application of a set of loads, is respectively

$$W_i = \frac{1}{2}\frac{S^2L}{AE} \quad \text{and} \quad W_i = \frac{1}{2}\int_0^L \frac{M^2dx}{EI}.$$

* After the discoverer, Alberto Castigliano (1847–1884), a distinguished Italian engineer. His "Théorie de l'équilibre des systèmes élastiques," (1879) is one of the pioneer works in theory of structures.

† The method can be extended without difficulty to include the case of yielding supports.

(A is assumed constant)

$$\therefore \frac{\partial W_i}{\partial P} = \frac{SL}{AE} \cdot \frac{\partial S}{\partial P},$$

for the case of direct stress, and

$$\frac{\partial W_i}{\partial P} = \int_0^L \frac{Mdx}{EI} \cdot \frac{\partial M}{\partial P},$$

for the case of bending. (The integration is with respect to x; hence the differentiation under the integral sign with respect to P is permissible.)

Now, if a bar is subjected to the action of several loads, of which P_r is one, we may always write

$$S = \bar{S} + P_r \cdot u_r$$

where $\bar{S}$ = stress due to all loads excluding P_r,
and u_r = stress due to load unity applied in the line of action of P_r.

Also $$M = \bar{M} + P_r m_r,$$

where $\bar{M}$ and m_r are defined in a similar manner.

We then have

$$\frac{\partial S}{\partial P_r} = u_r \quad \text{and} \quad \frac{\partial M}{\partial P_r} = m_r,$$

and the expressions for the deflection of a girder or a frame obtained by means of the derivative of the internal work with respect to a load at the point of deflection become identical with the previous Equations derived from the dummy unit loading.

SECTION II.—SPECIAL METHODS

C. Moment Area Method

17. First Principle.—Given the beam of Fig. 23; required the angular change in the elastic line between the points A and B' due to any loading. We have

$$\alpha_{B'} = \int_{B'}^{A} \frac{Mdx}{EI} \cdot m,$$

where m is the moment at any section distant x from B' due to unit couple applied at B'. Therefore

$$\alpha_{B'} = \int_{B'}^{A} \frac{Mdx}{EI}, \qquad (21)$$

since m equals unity at all points.

Referring to Fig. 23, the above expression obviously represents

numerically the area of the $\frac{M}{EI}$ diagram between A and B'. Now, if we wish to find the angular displacement between two tangents, M and N in *any* bent beam, we may for the purpose in hand view one of the points as a fixed end and find the relative rotation at the other point by the above method. We thus arrive at the general principle:

"In any bent beam the change in angle between any two points on the elastic line of the beam is numerically equal to the area of the $\frac{M}{EI}$ diagram between these two points."

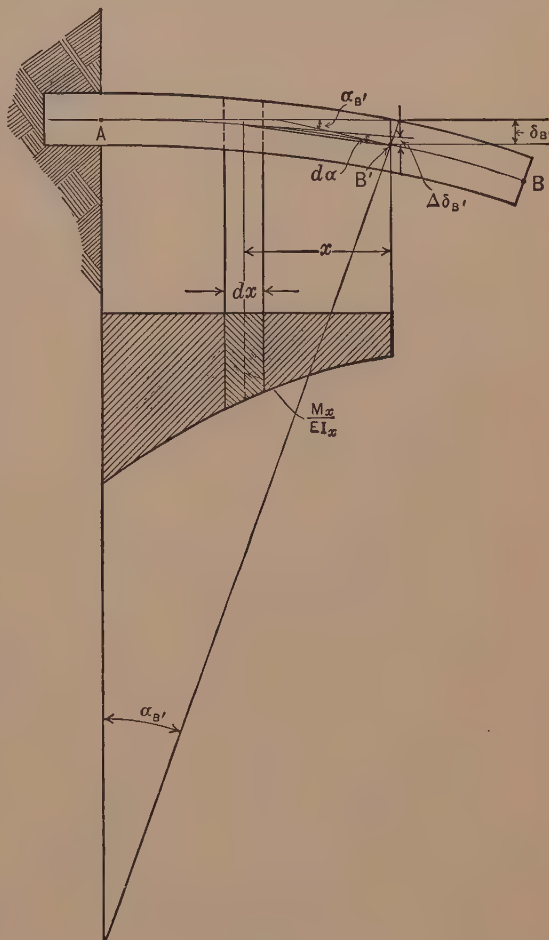


FIG. 23

18. Second Principle.—If it be required to find the vertical deflection of B' measured from a horizontal tangent at A, we have

$$\delta_{B'} = \int_{B'}^{A} \frac{Mdx}{EI} \cdot m,$$

where m is the moment at any section distant x from B' due to a vertical load of unity acting at B', whence

$$\delta_{B'} = \int_{B'}^{A} \frac{Mdx}{EI} \cdot x. \quad \text{. . . . .} \quad (22)$$

This expression is clearly equal, numerically, to the statical moment of the $\frac{M}{EI}$ diagram taken about a vertical through B'. Evidently this

proposition applies to the linear displacement of a given point from a tangent at some other point in *any* bent beam. This second general principle may be stated:

"The deflection of a point in any bent beam from a tangent at some other arbitrarily selected point is numerically equal to the statical moment of the $\frac{M}{EI}$ area between the two points, with respect to a line (normal to the reference tangent) through the deflected point."

These two very important propositions form the basis of what is commonly called the method of moment areas.*

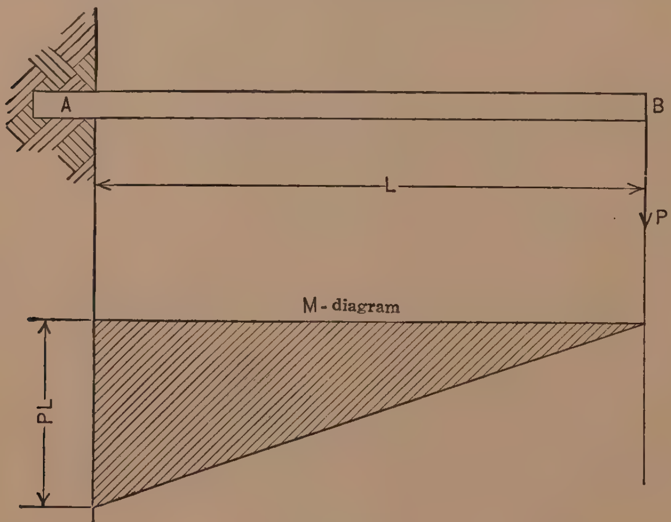


Fig. 24

The following examples will illustrate the manner of application of the principle.

Problem 1.—*Cantilever with load at end* (*I constant*) (Fig. 24).

We may write at once from preceding principles:

$$\alpha_B = \frac{PL^2}{2EI}; \quad \delta_B = \frac{PL^2}{2EI} \cdot \frac{2L}{3} = \frac{PL^3}{3EI}.$$

Problem 2.—*Cantilever with uniform load* (*I constant*) (Fig. 25).

From known properties of the parabola (see Table I) the area of moment diagram $= \frac{1}{6}wL^3$ and its centroid is $\frac{3}{4}L$ from the free end. Hence

$$\alpha_B = \frac{wL^3}{6EI}; \quad \text{and} \quad \delta_B = \frac{wL^4}{8EI}.$$

* The development of the moment area method as here defined is due to the late Professor Charles E. Greene of the University of Michigan (1874).

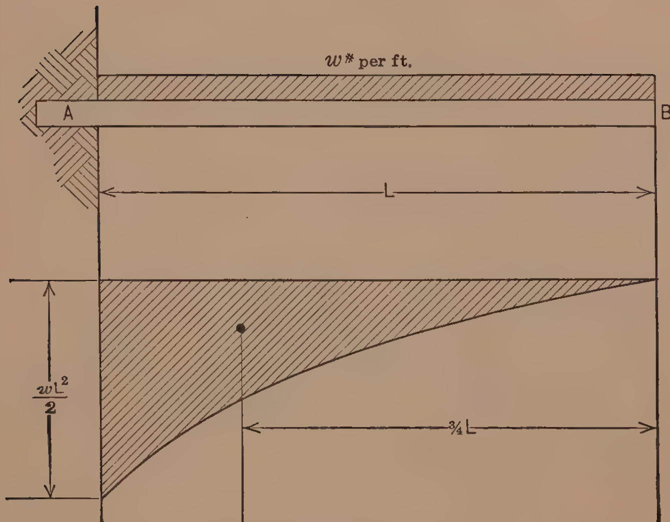


Fig. 25

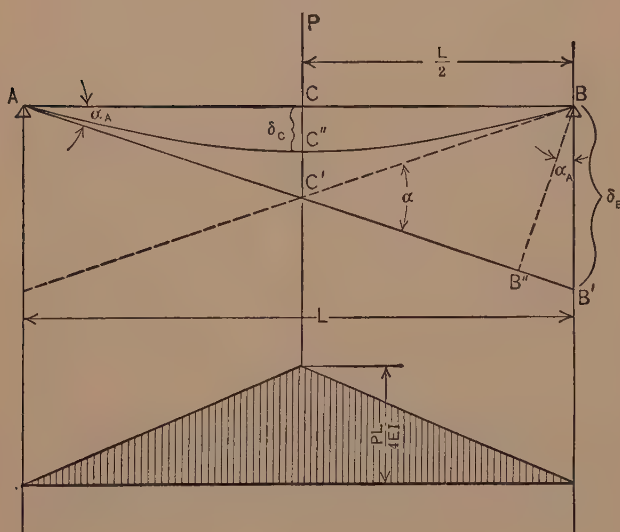


Fig. 26

Problem 3.—*Simple beam with load at center* (Fig. 26). We have at once

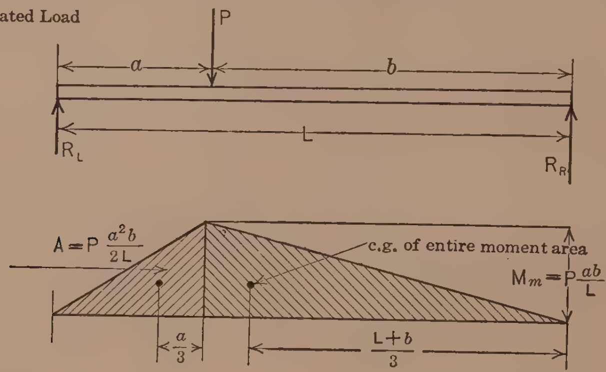
$$\alpha = \text{total area of } \frac{M}{EI} \text{ diagram} = \frac{PL^2}{8EI},$$

$$\alpha_A = \alpha_B = \frac{PL^2}{16EI}.$$

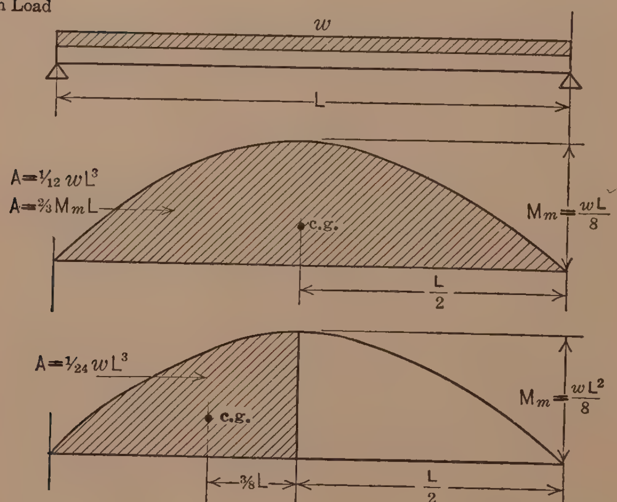
TABLE I

Properties of Certain Moment Diagrams Frequently Encountered

Concentrated Load



Uniform Load



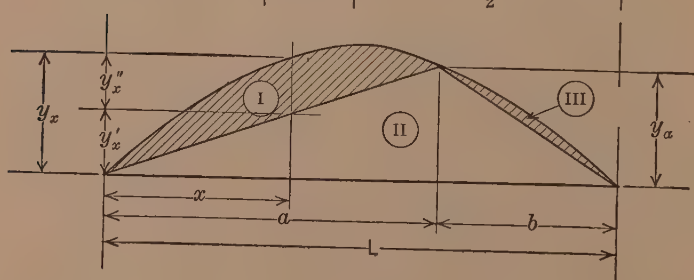


TABLE I—*Continued*

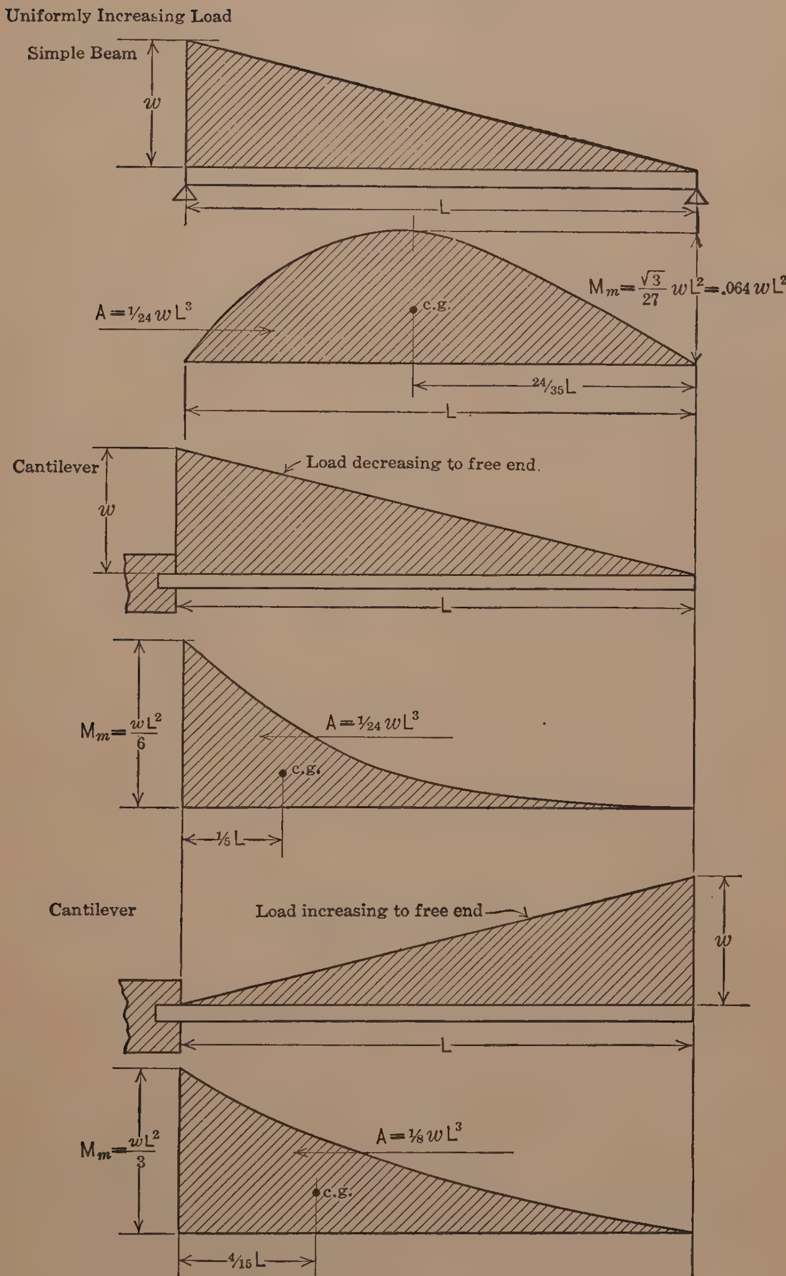
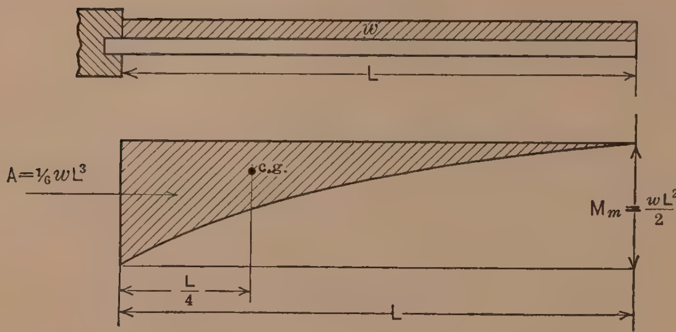


TABLE I—*Continued*



NOTE ON DIAGRAM (4), page 44.—$y_x = \frac{w}{2}(Lx - x^2)$; $y_x'' = y_x - y_x' = \frac{w}{2}(Lx - x^2) - \frac{x}{a} \cdot \frac{w}{2}a(L - a) = \frac{w}{2}(ax - x^2)$; i.e. the area (I) is identical with the moment curve of a simple beam of span "a." Similarly (II) is identical with the moment curve for span "b." Since this relation holds for any pair of values of "a" and "b," the division of the moment area as shown above may always be used to obtain the statical moments about any point.

To obtain δ_c, some special consideration is necessary. The moment area method gives the deflection of any point *from a tangent* at some other point; in this problem the desired deflection δ_c is from the original position of the beam, and the moment area method does not give this directly. In such case we may proceed as follows: Since α_A is very small, $BB' = BB'' = \delta_B =$ (from the second moment area principle) $\frac{PL^2}{8EI} \cdot \frac{L}{2}$. From the geometry of the figure, $CC' = \frac{\delta_B}{2} = \frac{PL^3}{32EI}$. Also, $CC' - C'C'' = CC'' = \delta_c$. But $C'C''$ (from second moment area principle) $= \frac{PL^2}{16EI} \cdot \frac{L}{6}$, whence $\delta_c = \frac{PL^3}{48EI}$. The same general method will apply to finding any simple beam deflection by means of moment areas.

19. Independent Derivation.—The moment area method may be derived quite simply without recourse to the philosophy of the work of deformation. For it is clear, Fig. 23, that the total angular change between the faces of the beam at A and at B' must be the summation of the angular changes of all the elementary sections dx lying between these points. But we have shown (page 17) that the angular change $d\alpha$ between the two faces of an elementary section is $\frac{Mdx}{EI}$; therefore

the total change between A and B is $\int_A^{B'} \frac{Mdx}{EI}$ = area of $\frac{M}{EI}$ diagram between A and B. This establishes the first moment area proposition.

To derive the second principle, we note that the contribution of the distortion of an elementary section dx to the deflection at B'

$$= \Delta\delta_{B'} = d\alpha \cdot x = \frac{Mdx}{EI} \cdot x; \quad \text{and} \quad \delta_{B'} = \int_A^{B'} \frac{Mdx}{EI} \cdot x$$

= statical moment of $\frac{M}{EI}$ diagram between sections A and B' about B'.

The moment area method furnishes a general method of attack on all beam deflection problems, and many types of rigid frames can be analyzed advantageously by its use. By means of Table I used in conjunction with this method, a variety of deflection results may be written out at once, and the tedious integration processes of the general method of work or the method based on the differential equation of the elastic line are thus avoided. The student will be well repaid for taking time to thoroughly master the principle.

D. Method of Elastic Weights

20. Simple Beams.—If we examine the fundamental formula

$$\delta = \int_A^B \frac{Mmdx}{EI},$$

we note that it may be approximately evaluated as follows (see Fig. 27): Construct the $\frac{M}{EI}$ diagram and divide it into a convenient number of small strips Δx; construct the m diagram, determining the ordinates corresponding to the centers of the strips Δx. Then evidently

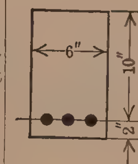
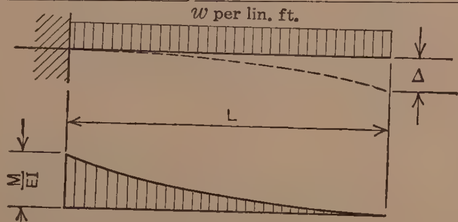
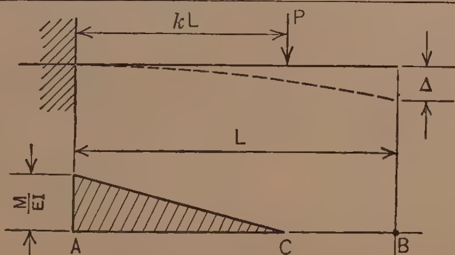
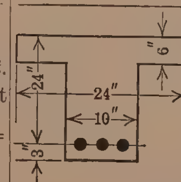
$$\delta_q = \text{(approximately)} \sum \frac{M \cdot \Delta x}{EI} \cdot m. \quad . \quad . \quad . \quad . \quad (23)$$

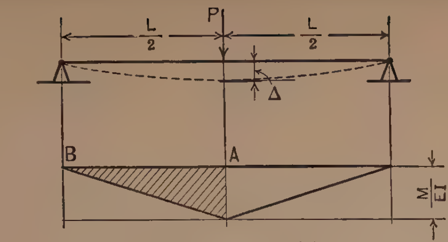
Now let us imagine the same beam loaded with a varying load, w per foot. An approximate value for M_q may be obtained as follows: Construct the influence line for the moment at q (Fig. 28). Divide the distributed load w into a series of concentrations $w \cdot \Delta x$, Δx being any convenient small distance; the smaller it is taken the more accurate the approximation. Then if M_{1q} is the ordinate to the influence line (taken in each case to correspond with the center of the space Δx)

$$M_q = \text{(approximately)} \Sigma w \cdot \Delta x \cdot M_{1q}. \quad . \quad . \quad . \quad . \quad (24)$$

TABLE I*a*

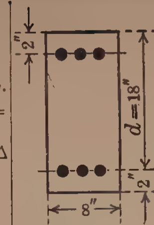
DEFLECTION OF BEAMS (WITH SPECIAL REFERENCE TO REINFORCED CONCRETE BEAMS)

Loading, End Conditions and $\frac{M}{EI}$ Diagram	Derivation of Formulæ	Illustrative Calculations for Reinforced Concrete Beams
	I *Cantilever beam with load P at end.* $M = PL$. ∴ Moment of area AB about $B = \Delta = \frac{ML^2}{3EI} = \frac{PL^3}{3EI}$ or $\frac{(f_s + nf_c)L^2}{3dE_s} =$ Δ for reinforced concrete beams.	*Given:*— $f_c = 650 \quad f_s = 16000$ $n = 15 \quad L = 8' - 0''$ $\Delta = \frac{16{,}000 + 15(650)}{3(10)30{,}000{,}000}(96)^2$ $\Delta = .264''$
	II *Cantilever beam with uniform load* $M = \frac{wL^2}{2}$. ∴ Moment of area AB about $B = \Delta = \frac{ML^2}{4EI} = \frac{wL^4}{8EI}$ or $\frac{(f_s + nf_c)L^2}{4dE_s} =$ Δ for reinforced concrete beams.	For same cross-section, span and maximum stresses as case I, $\Delta = \frac{3}{4}(.264) = .198''$
	III *Cantilever beam with load P at any point.* $M = PkL$. ∴ Moment of area AC about $B = \Delta = \frac{MkL}{2EI}\left[\frac{2}{3}kL + (1 - k)L\right] = \frac{PL^3}{3EI}[3k^2 - k^3]$ or for reinforced concrete beams $\Delta = \frac{(f_s + nf_c)}{6dE_s}L^2(3k - k^2)$.	*Given:*— $f_c = 713 \quad f_s = 17{,}500$ $n = 13.6 \quad L = 22' - 6''$ $k = .75 (3k - k^2) = 1.688$ $\Delta = \left[\frac{17{,}500 + 13.6(713)}{6(24)30{,}000{,}000}\right](270)^2 1.688$ $\Delta = .774''$



IV

Simple beam load P at center. $M = \frac{1}{4}PL$. $\therefore$ Moment of area AB about $B = \Delta =$ $\frac{M}{3EI}\left(\frac{L}{2}\right)^2 = \frac{1}{48}\frac{PL^3}{EI}$, or $\frac{(f_s + nf_c)}{12dE_s}L^2 = \Delta$ for reinforced concrete beams.



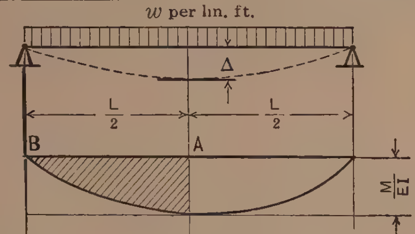
Given:—

$f_c = 750 \quad f_s = 18000$

$n = 12 \quad L = 16'\text{-}0''$

$$\Delta = \frac{18{,}000 + 12(750)}{12 \times 18 \times 30{,}000{,}000}(192)^2$$

$\Delta = .153''$



V

Simple beam uniformly loaded. $M = \frac{wL^2}{8}$. $\therefore$ Moment of area AB about $B = \Delta$

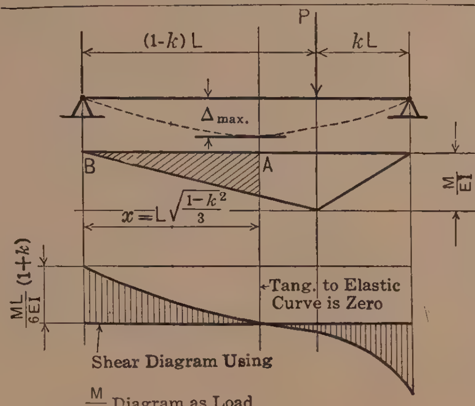
$$= \frac{2}{3}\left(\frac{ML}{2EI}\right)\left(\frac{5L}{16}\right) = \frac{5}{384}\frac{wL^4}{EI} \text{ or}$$

$$\frac{f_s + nf_c}{dE_s}\left(\frac{5}{48}L^2\right) = \Delta$$

for reinforced concrete beams.

For same cross-section, span and maximum stresses as in preceding case IV.

$$\Delta = 12\left(\frac{5}{48}\right).153 = .192''$$



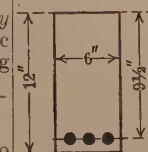
VI

Simple beam—concentrated load at any point. Δ_{max} occurs where tangent to elastic curve = 0, i.e., where shear due to loading with $\frac{M}{EI}$ is zero. Shear $= \frac{ML}{6EI}(1 + k) - \frac{Mx^2}{2EI(1-k)L}$, whence $x = L\sqrt{\frac{1-k^2}{3}}$. Also

$$M_x = M\frac{\sqrt{\frac{1-k^2}{3}}}{1-k} \text{ and } M = P(k - k^2)L,$$

whence $\Delta_{max} = \frac{M_x x^2}{3EI} = \frac{PL^3}{9EI}(k - k^3)\sqrt{\frac{1-k^2}{3}}$, or for reinforced concrete beams

$$\Delta_{max} = \sqrt{\frac{1-k^2}{3}}\left(\frac{f_s + nf_c}{9dE_s}\right)(1 + k)L^2.$$



Given:—

$f_c = 700$

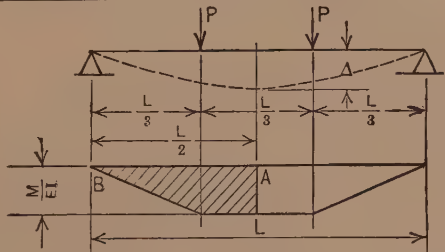
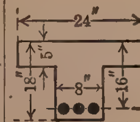
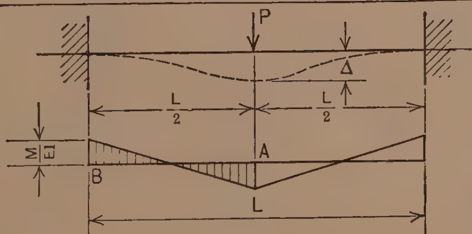
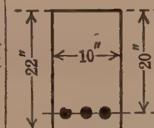
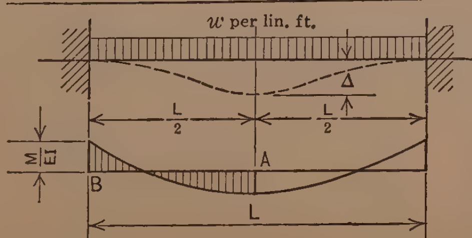
$f_s = 15{,}000$

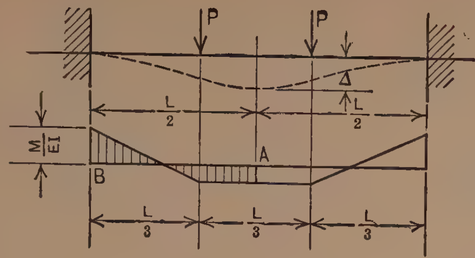
$n = 15 \; L = 13'\text{-}0'' \; k = .3$

$$\Delta = \sqrt{\frac{1 - (.3)^2}{3}}\left[\frac{15{,}000 + 15(700)}{9(9\frac{1}{2})30{,}000{,}000}\right](1 + .3)(156)^2$$

$\therefore \Delta = .170''$

TABLE Ia—*Continued*

Loading, End Conditions and $\frac{M}{EI}$ Diagram	Derivation of Formulæ	Illustrative Calculations for Reinforced Concrete Beams
	VII *Simple beam—equal concentrated loads. Pc $\frac{1}{3}$ points.* $M = \frac{1}{3}PL$ and moment of area AB about $B = \frac{5L}{12}\left(\frac{ML}{6EI}\right) + \frac{2L}{9}\left(\frac{ML}{6EI}\right)$ $= \frac{23}{216}\frac{ML^2}{EI} = \frac{23}{648}\frac{PL^3}{EI}$ or $\left(\frac{nf_c + f_s}{dE_s}\right)\frac{23}{216}L^2$ $= \Delta$ for reinforced concrete beams.	 *Given:*— $f_c = 323$ $f_s = 18{,}000$ $n = 15 \quad L = 20'\text{-}0''$ $\Delta = \left[\frac{15(323) + 18{,}000}{16 \times 30{,}000{,}000}\right]\frac{23}{216}(240)^2 = .292''$
	VIII *Fully restrained beam with Pc center.* $M = \frac{1}{8}PL$ and moment of area AB about $B = \frac{2M\frac{L}{2}}{2}\left(\frac{L}{3}\right) - M\frac{L}{2}\left(\frac{L}{4}\right) = \frac{1}{24}\frac{ML^2}{EI} =$ $\frac{1}{192}\frac{PL^3}{EI}$. $\therefore \frac{1}{24}\left(\frac{nf_c + f_s}{dE_s}\right)L^2 = \Delta$ for reinforced concrete beam.	 *Given:*— $f_c = 650$ $f_s = 18{,}000$ $n = 15 \quad L = 20'\text{-}0''$ $\Delta = \left[\frac{15(650) + 18{,}000}{20 \times 30{,}000{,}000}\right]\frac{(240)^2}{24}$ $\Delta = .111''$
	IX *Fully restrained beam with uniform load w.* $M = \frac{1}{12}wL^2$ and moment of area AB about $B = \frac{2}{3}\left(\frac{L}{2}\right)\frac{3M}{2EI}\left(\frac{5L}{16}\right) - \frac{L}{2}\left(\frac{M}{EI}\right)\frac{L}{4} =$ $\frac{1}{32}\frac{ML^2}{EI} = \frac{1}{384}\frac{wL^4}{EI}$. $\therefore \frac{1}{32}\left(\frac{nf_c + f_s}{dE_s}\right)L^2 =$ Δ for reinforced concrete beam.	For same stresses, span, cross-section as in case VIII $\Delta = \frac{24}{32}(.111) = .0833$



X

Fully restrained beam with equal loads Pc $\frac{1}{3}$ points. $M = \frac{2PL}{9}$ and moment of area AB about $B = \frac{23}{216}\left(\frac{3ML^2}{2EI}\right) - \frac{ML}{2EI}\left(\frac{L}{4}\right) = \frac{5}{144}\frac{ML^2}{EI} = \frac{5}{648}\frac{PL^3}{EI}$.

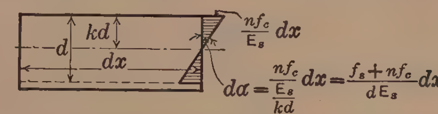
$\therefore \frac{1}{28.8}\left(\frac{nf_c + f_s}{dE_s}\right)L^2 = \Delta$ for reinforced concrete beam (see case VII).

For same stresses, span and cross-section as in case VIII

$$\Delta = \frac{24}{28.8}(.111) = .0925$$

Note on Deflections of Reinforced Concrete Beams

The formula—$\Delta_{max} = C\frac{L^2}{dE_s}(f_s + nf_c)$ (C being a constant) is known as Maney's formula (see Proc. Am. Soc. of Testing Materials, Vol. XIV, for full discussion). It will be noted that the formula for reinforced concrete beams is obtained by substituting $\frac{f_s + nf_c}{dE_s}$ for $\frac{M}{EI}$ in the general formula. This may be justified as follows:—assuming linear distribution of stress over any section, we must have from the geometry of the strained beam . . . $d\alpha = \frac{\frac{f_c}{E_c}dx}{kd} = \frac{\frac{nf_c}{E_s}dx}{\left(1+\frac{f_s}{nf_c}\right)d} = \frac{f_s + nf_c}{dE_s}dx = \frac{Mdx}{EI}$.



It will be noted that where loading, span, depth and maximum stresses are the same, Δ_{max} is the same whether the beam be a T-beam, a doubly reinforced or a simple rectangular beam. "d" is always measured from compression face to center of tensile steel. Usually f_s and f_c are known or can be easily computed. The stresses are assumed to vary as the M-diagram. Many tests verify this formula.

But the influence line for M_q is *numerically* precisely the same thing as the moment diagram for the beam due to unity at q; in other words $M_1 = m$. Now if w should equal $\frac{M}{EI}$, it is clear that the expres-

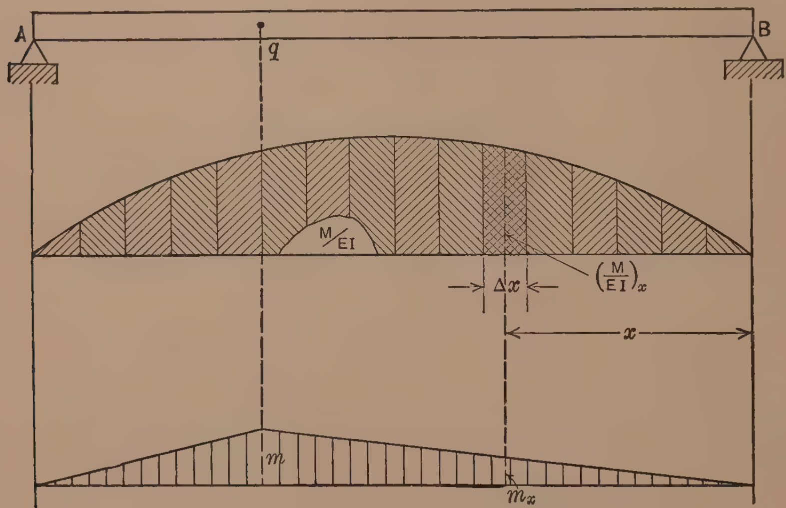


FIG. 27

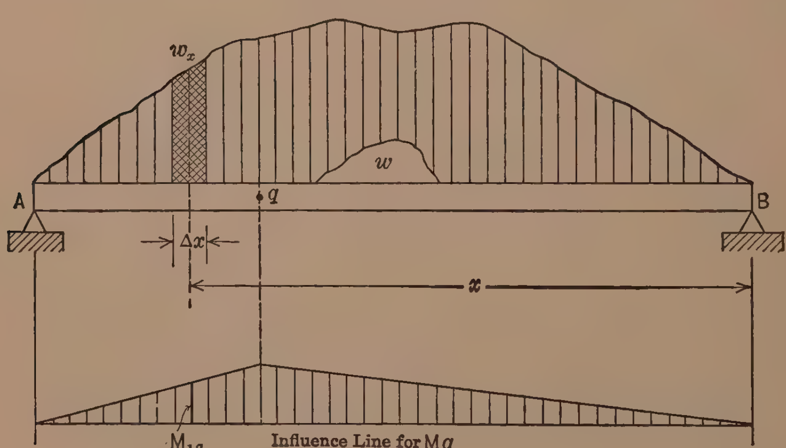


FIG. 28

sions (23) and (24) are *numerically identical*; i.e., the deflection of any given point q in a simple beam AB is obtained by applying to the beam the *actual* $\frac{M}{EI}$ diagram *as an imagined load curve*, and computing the moment at q. This fictitious moment is *numerically* equal to the actual

deflection. Since this is true of all points in the beam, a moment diagram constructed for the imagined loading of $\frac{M}{EI}$ per foot is identical with the actual elastic line.

Similarly

$$\alpha_q = \int \frac{M\,dx}{EI} \cdot m,$$

where m = moment at any section due to a *couple* of unity acting at q. Drawing a curve for m (Fig. 29), we note that it is identical numerically with the *shear influence line* for section q. Hence we deduce that the

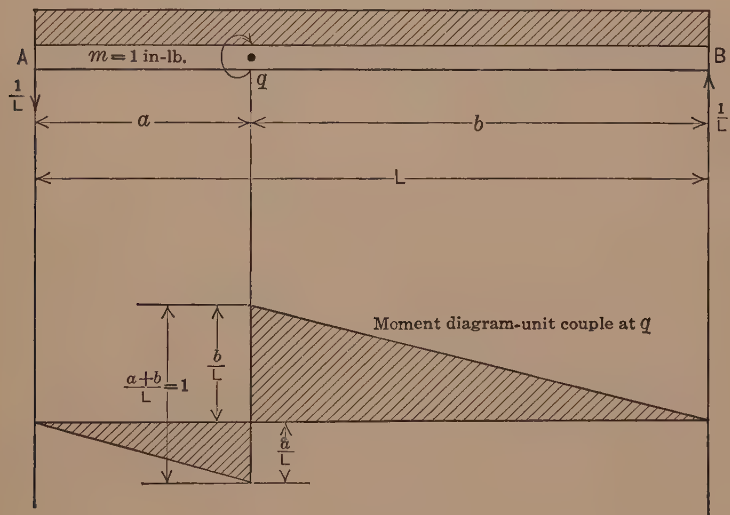


FIG. 29

angular change at any section of a simple beam AB is equal to the shear at the section due to an imagined loading equivalent to the $\frac{M}{EI}$ curve.

21. Graphical Representation of Elastic Curve as a String Polygon.— From the well-known relations between the moment diagram and the equilibrium polygon we may construct the elastic line of a beam according to the above method, by a strictly graphical process. We first lay off a force polygon of the actual loads and, taking any convenient pole distance H_{I}, draw a string polygon. (See Fig. 30.) If y_{I} is an ordinate to the polygon, $H_{\text{I}} y_{\text{I}} = M$. Next, lay off on the base of the string polygon convenient small divisions Δx and treat the small areas

$$\frac{M \Delta x}{EI} = \frac{H_{\text{I}} y_{\text{I}} \Delta x}{EI},$$

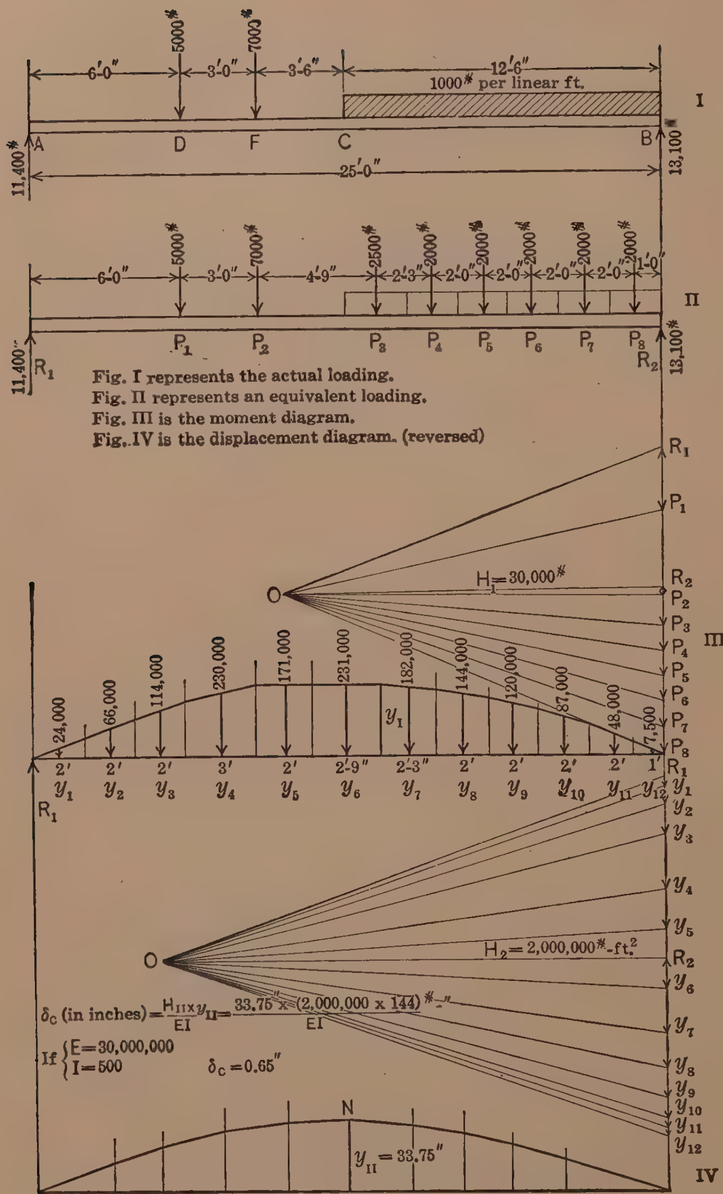

5000#
7000#
6'0"
3'0"
3'6"
12'6"
1000# per linear ft.
I
A
D
F
C
B
11,400#
13,100#
25'0"
2500#
2000#
4'9"
2'3"
2'0"
1'0"
II
P_1 P_2 P_3 P_4 P_5 P_6 P_7 P_8
R_1
R_2
Fig. I represents the actual loading.
Fig. II represents an equivalent loading.
Fig. III is the moment diagram.
Fig. IV is the displacement diagram. (reversed)
O
$H_1 = 30,000\#$
III
24,000
66,000
114,000
230,000
171,000
231,000
182,000
144,000
120,000
87,000
48,000
7,500
y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9 y_{10} y_{11} y_{12}
2' 2' 2' 3' 2' 2'9" 2'3" 2' 2' 2' 2' 1'
$H_2 = 2,000,000\#\text{-ft.}^2$
δ_C (in inches) $= \frac{H_{II} \times y_{II}}{EI} = \frac{33.75'' \times (2,000,000 \times 144)\ \#\text{-}''}{EI}$
If $\begin{cases} E = 30,000,000 \\ I = 500 \end{cases}$ $\delta_C = 0.65''$
N
$y_{II} = 33.75''$
C
IV

FIG. 30

as loads, and with any pole distance H_{II} draw a second string polygon. Any ordinate y_{II} of this polygon will, if multiplied by H_{II}, equal the moment at the point where the ordinate is drawn due to a loading of $\frac{M}{EI}$ per unit of length, and hence will numerically equal the deflection.

21a. Examples.—A few simple illustrations will make clear the method of application of the principle of elastic loads.

Problem 1.—Fig. 30*a*—*Simple beam with load at center; to find* α_q *and* δ_q.

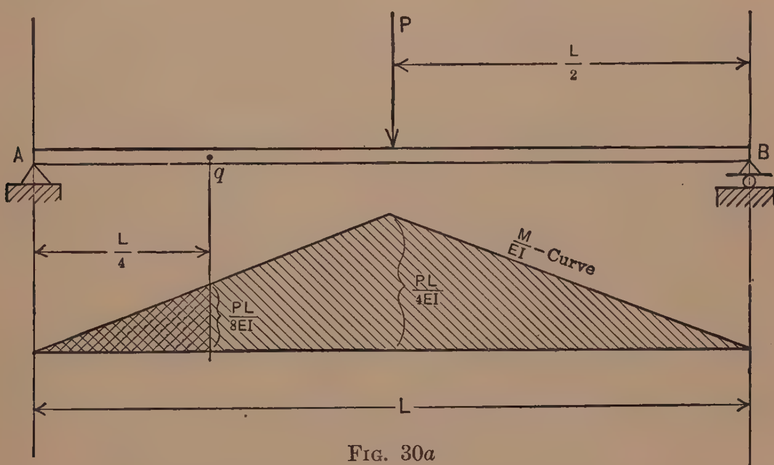


Fig. 30*a*

We have seen that α_q is numerically equal to the shear at q in the beam when loaded with the $\frac{M}{EI}$ diagram. Therefore,

$$\alpha_q = \frac{PL^2}{16EI} - \frac{PL^2}{64EI} = \frac{3PL^2}{64EI}.$$

Also, δ_q is numerically equal to the bending moment at q in the beam when loaded with the $\frac{M}{EI}$ diagram.

Therefore,

$$\delta_q = \frac{PL^2}{16EI} \times \frac{L}{4} - \frac{PL^2}{64EI} \times \frac{L}{12} = \frac{11PL^3}{768EI}.$$

Problem 2.—Fig. 30*b*—*Simple beam uniformly loaded; to find* α_q and δ_q.

Since the area of the $\frac{M}{EI}$ curve $FH'G = \frac{2}{3}(HH')\cdot L = \frac{wL^3}{12EI}$, the reaction at A due to the $\frac{M}{EI}$ loading is $R'_A = \frac{wL^3}{24EI}$.

The area

$$FKK' = \frac{w}{EI}\int_0^{\frac{L}{4}} \left(\frac{Lx}{2} - \frac{x^2}{2}\right)dx = \frac{5}{384}\,\frac{wL^3}{EI}.$$

Hence

$$\alpha_q = \text{Shear at } q = \frac{wL^3}{24EI} - \frac{5}{384}\,\frac{wL^3}{EI} = \frac{11}{384}\,\frac{wL^3}{EI}.$$

Likewise, δ_q = bending moment at q due to $\frac{M}{EI}$ loading

$$= R'_A \times \frac{L}{4} - \text{moment of } FKK' \text{ about } q$$

$$= \frac{wL^4}{96EI} - \frac{w}{EI}\int_0^{\frac{L}{4}} \left(\frac{Lx}{2} - \frac{wx^2}{2}\right)\left(\frac{L}{4} - x\right)dx = \frac{57}{6144}\,\frac{wL^4}{EI}.$$

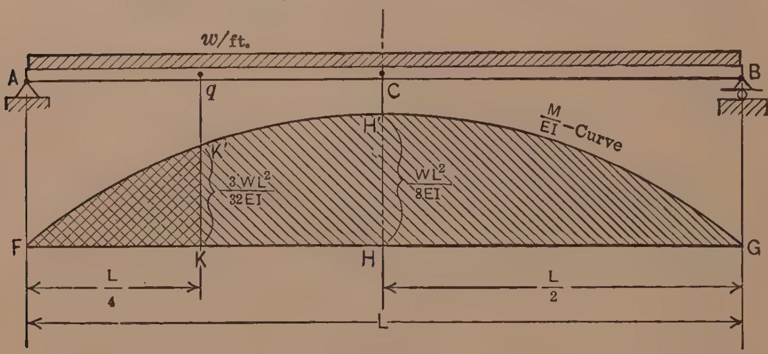


Fig. 30*b*

Problem 3.—*Simple beam with partial uniform load and concentrated loads; to find elastic curve graphically.*

Solution is completely shown in Fig. 30.

22. Application to Beams not Simply Supported.—The method of elastic loads as developed in the preceding articles applies only to beams simply supported at the ends. (It will be remembered that the deduction was based upon the numerical identity of the m_q diagram and the moment influence line for q,—a relation which holds only for a simple beam.) The method can be generalized to apply to all types of beams, but since we shall make little use of the method in any but the simple beam case, the general method will not be developed here.*

* For a luminous account of the general theory of representation of the elastic line of *any* beam as a moment diagram for a suitably chosen "substitute" beam and suitably chosen loads, see a paper by Professor H. M. Westergaard, "Deflection of Beams by the Conjugate Beam Method," Journal of Western Society of Engineers, Nov. 1921.

The student should note that the moment area method gives *directly* change of angle between tangents at two separated points and deflection from tangents, while loading with the $\frac{M}{EI}$ diagram gives *directly* the angular and linear displacement referred to the original position. The former is therefore the readier method in dealing with cantilevers and the latter with simple beams.

As will be seen from the last problem under Art. 18, however, the moment area principle is easily adapted to the simple beam case, even when deflections from a tangent are not directly desired. As regards the

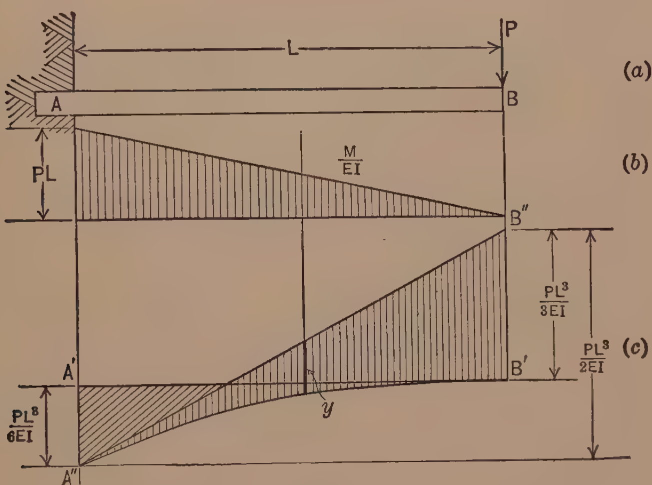


FIG. 31

application of the elastic load method to cantilever beams, it is evident that any cantilever may be regarded as one-half of a symmetrical simple beam, suitably loaded, and the method may thus be very simply extended to cover this case. It may also be of interest to the student to show that if a cantilever AB (Fig. 31a) is directly loaded with the $\frac{M}{EI}$ diagram (Fig. 31b) and the corresponding moment curve drawn ($A''B'$—Fig. 31c), then the true deflection line will be determined by the ordinate y to the curve $A''B'$, *measured from the base* $A''B''$. ($A''B''$ is tangent to $A''B'$ at A''.)

The method of loading with the $\frac{M}{EI}$ diagram together with the two principles enunciated in Arts. 17 and 18 are grouped by many writers

under title of the moment area method.* So far as the treatment of beams goes, the designation is apt enough; both make use of the $\frac{M}{EI}$ diagram in a very similar manner so far as practical detail is concerned. But the underlying conceptions of the two methods are quite different, as will be clear from the preceding pages. Furthermore, the notions involved in the procedure of treating the $\frac{M}{EI}$ diagram as a load curve are identical with those involved in the treatment of truss deflections in Art. 24, and for this case the designation of "moment area" seems hardly suitable; it is almost universally termed the method of elastic weights.

23. Advantages as Compared to General Method.—The same remarks apply here as were made in Art. 18 regarding the moment area method. The two methods taken together, each supplementing the other, constitute an analytical tool of far-reaching practical importance in the treatment of deflections and therefore in the analysis of statically indeterminate structures. Their use obviates all necessity of formal integration in most practical cases, and while the integrals involved in the work equations are of the simplest kind, their evaluation is tedious and time-consuming, and is a common source of error. In most deflection problems, whether the loading results in a simple and easily expressible moment curve or a complex and irregular one, the moment area or elastic weight method is likely to prove by far the simplest working method for finding the deflections. Table I will greatly facilitate the work.

It should also be mentioned that the relationships brought out by the above principles (e.g., the fact that for a beam or series of beams with ends fixed, the positive and negative $\frac{M}{EI}$ areas must balance) are frequently of importance in the analysis and checking of problems where the principles are not used to obtain numerical results.

24. Truss Deflections.—The principle of elastic weights can easily be extended to the case of truss deflections. In Fig. 32 let us examine the deflection of the truss due to (1) the deformation of the chord member BC and (2) the web member Cc.

* The method of representing the deflection line as a string polygon and the concomitant method of computing individual deflections as bending moments due to a fictitious loading are due to O. Mohr, "Beitrag zur Theorie der Holz- und Eisen Konstruktionen," Zeitschrift des Architekten und Ingenieur Vereines zu Hannover, 1868.

For BC

$$\delta = \frac{S_{BC} u_{BC} L_{BC}}{A_{BC} E},$$

or omitting subscripts $= \frac{SL}{AE} \cdot u$. $\frac{SL}{AE}$ is a constant for any given loading. The deflection of a specific point, as e, is obtained by multiplying this constant by u_{BC-e}, the stress in BC due to unity at e. For the deflection of d or c the deflection constant is the stress in BC due to unity at d or c. A little reflection will make it clear that the u diagram

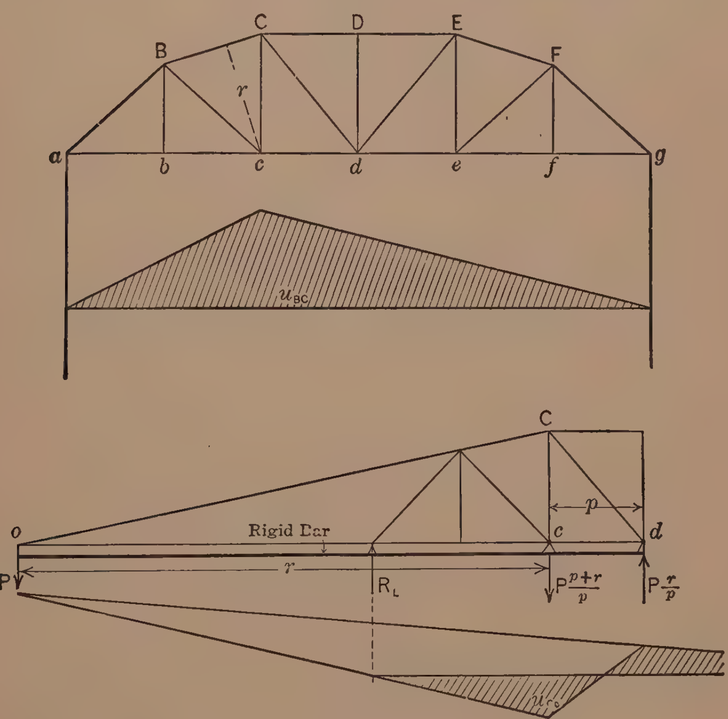


FIG. 32

for BC is, *to some scale*, the deflection diagram for a change of length in BC. But the u diagram is identical with the ordinary influence line for the stress in BC, and is obtained by placing a unit load at the center of moments for BC and drawing the moment diagram divided by r (since $u = \frac{m}{r}$). If instead of unity we apply a load of $\frac{SL}{rAE} = \frac{\Delta L}{r}$ then the moment diagram is numerically exactly equivalent to the true deflec-

tion line. The same rule holds for all chord members; hence, to construct the deflection diagram for a truss due to the deformation of the chord members, *load the moment center for each chord with* $\frac{\Delta L}{r}$ *and draw the moment diagram.*

For Cc a similar line of reasoning leads to the conclusion that the influence line for the stress in Cc is, to some scale, the deflection curve of the truss for a change of length in Cc. For chord members it is clear on inspection that the influence line is numerically the same as the moment diagram for unity placed at the center of moments for chord. If now we imagine a vertical load applied to the truss at o through a rigid bar o-c-d connected to the truss at c and d only, we see that the moment diagram due to a load of $\frac{1}{r}$ at o is numerically the stress influence line for Cc, and if $\frac{\Delta L}{r}$ be applied at o, the resulting moment diagram is the true deflection line. Now a load P applied to the truss in the above manner at o is equivalent to loads $P\frac{r+p}{p}$ and $-\frac{Pr}{p}$ applied respectively at c and d as shown. Hence the law for the web members: To draw the deflection line for the truss due to a change of length in any web, apply to joints adjacent to section which is cut to find the stress in the web, the loads $\frac{\Delta L}{r}\cdot\frac{p+r}{p}$ and $-\frac{\Delta L}{r}\cdot\frac{r}{p}$. The resultant moment diagram is the actual deflection curve.

We thus have a general method of constructing the deflection line due to the distortions of all members, for any truss, by the method of elastic loads. The method is fully illustrated in Problem II, page 78.

It may be interesting to note a similarity between the methods for beam and truss. Beam deflections are computed for the effects of bending moment only and hence are analogous to truss deflections due to deformation of the chords. It will be recalled that the elastic load for each small section of the beam, $\frac{M\Delta s}{EI}$, is the angular change due to the distortion of the element Δs. It is evident that the elastic load for the truss, $\frac{\Delta L}{r}$, is also the angular change due to the distortion of the chord member. Hence the law is sometimes stated that the deflections due to bending in a beam or truss are obtained by loading the span with the numerical equivalent of the total *angular change* at each point.

E. The Williot Displacement Diagram

25. General Theory.—Any point, as C (Fig. 33), connected to points A and B by a pair of bars, AC, BC, can obviously be displaced only (a) by a shift in position of A or B, or (b) by change in length of one or both of the bars. Knowing the shift of A and B and the deformation of AB and BC, we easily locate the final position of C ($= C_2$ in figure) graphically by swinging arcs with the new locations of A and B as centers and the new lengths of AC and BC as radii to an intersection in C_2. In the truss of Fig. 34 we may apply this graphical method to obtain the deflections. To make the construction clear we shall assume the relative deformations all equal and equal to $\frac{1}{10}L$, positive or negative as indicated. The point A and the line Aa are fixed; a' is therefore easily located; with these points as centers and the deformed lengths of AB and aB as radii we strike arcs which will intersect in the final location of $B (= B')$; with B' and a as centers and the deformed lengths Bb and ab as radii we locate b' similarly; from B' and b' and the deformed lengths Bc and bc we locate c', and so on.

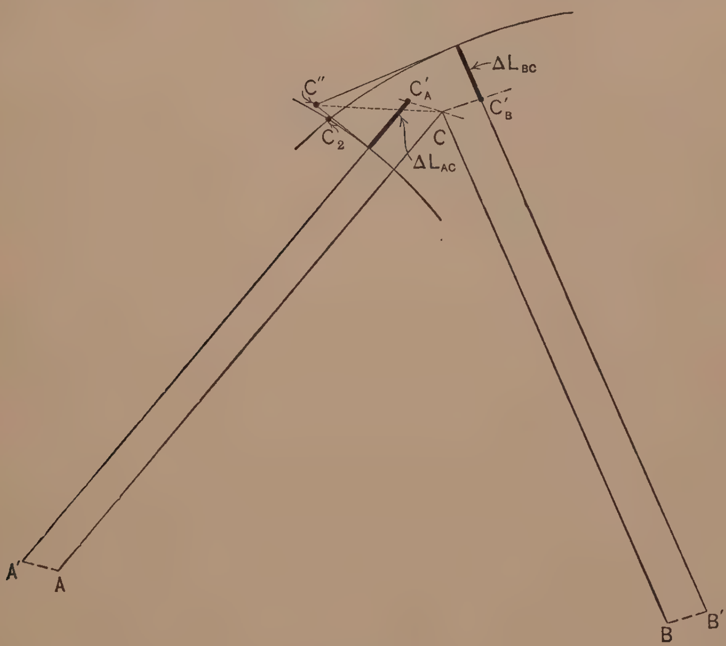
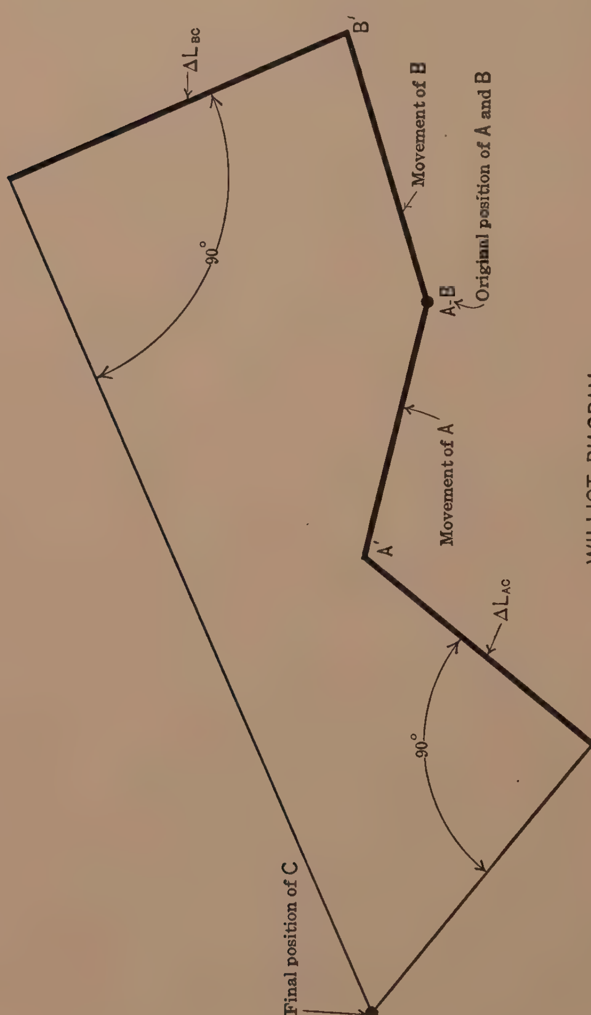


Fig. 33

This simple construction is, theoretically, always available for obtaining truss deflections when (as is nearly always the case) the truss is an assemblage of triangles. It is of little use as a working method,



WILLIOT DIAGRAM

FIG. 33*a*

however. We have noted that in the deflection of framed structures we are dealing with deformations and displacements which are exceedingly small compared with the lengths of members. The deformations

seldom exceed $\frac{1}{2000}L$, and in many deflection problems they are much less. To plot such quantities to any manageable scale on the same diagram with the frame itself is out of the question.

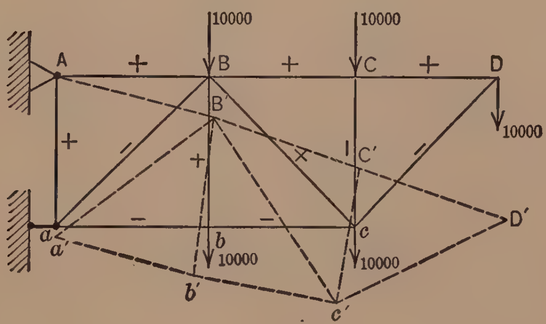


FIG. 34

TABLE A

Member	S, Lbs.	L, Ins.	A, Ins.^{2}	$\frac{L}{A}$	$E \cdot \Delta L = \frac{SL}{A}$	u_D* Lbs.	$\frac{SL}{A} \cdot u_D$*
AB	+90,000	180	6.0	30.0	+2,700,000	+3.0	8,100,000
$BC=CD$	+10,000	180	2.0	90.0	+ 900,000	+1.0	1,800,000
Dc	−14,100	253	2.0	126.5	−1,785,000	−1.41	2,520,000
$bc=ab$	−40,000	180	4.0	45.0	−1,800,000	−2.0	7,200,000
aA	+50,000	180	3.5	51.4	+2,570,000	+1.0	2,570,000
aB	−70,500	253	8.0	31.4	−2,230,000	−1.41	3,150,000
Bb	+10,000	180	1.0	180.0	+1,800,000		
Bc	+42,300	253	3.0	84.3	+3,560,000	+1.41	5,040,000
Cc	−10,000	180	2.0	90.0	− 900,000		

$E \cdot \delta_D = 30{,}380{,}000$
$\delta_D = 1.01''$

* The last two columns are added to give a check on the deflection at D. They are not required in the construction of the Williot diagram.

This fact of very small deformations, however, leads to a modified graphical method of the highest usefulness. For with deformations so small, the above described process of swinging arcs about such points as A and B may permissibly be replaced by erecting tangents at the

ends of the radii. (The student will best be convinced of this by attempting the exact construction in a simple example, say, $\Delta L = \frac{L}{2000}$. In the adjoining illustration (Fig. 33), the deformations are in the

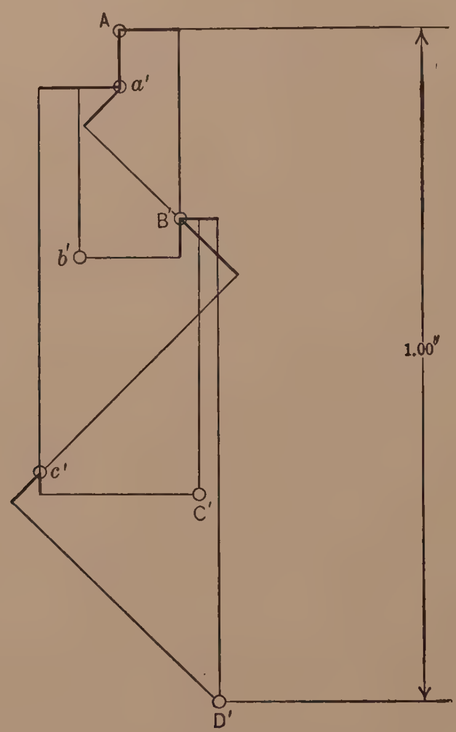


FIG. 34*a*

neighborhood of 200 times the natural size. In an actual truss the maximum change of length in a member 50 feet long would be little over $\frac{1}{4}$ inch.)

In Fig. 33 the exact construction gives the new location of C as C_2; the approximate construction, by means of perpendiculars erected at the ends of the radii, gives the displaced position as C''. Here the error is considerable, but for a relative deformation of $\frac{1}{100}$ of that shown, the difference would be practically negligible.

Following the detail of the approximate construction as shown in Fig. 33, we note that CC'_A and CC'_B are equal to AA' and BB'. These quantities must be known before the construction can be

started. Having these laid off, we next lay off ΔL_{A-C} from C'_A and ΔL_{B-C} from C'_B and from the extremities of these lines (which are the ends of the members after deformation) we erect perpendiculars and prolong to their intersection instead swinging arcs about A' and B'. This last step is the keynote of the construction, which, it will be observed, may be carried through *without any knowledge of the actual lengths AC and BC*. Since we make no use of these quantities, we may draw the displacement diagram to any scale we please, quite independently of the framework. It is so shown in Fig. 33*a*.

This construction is known as the Williot displacement diagram, after the French engineer Williot, by whom it was developed.

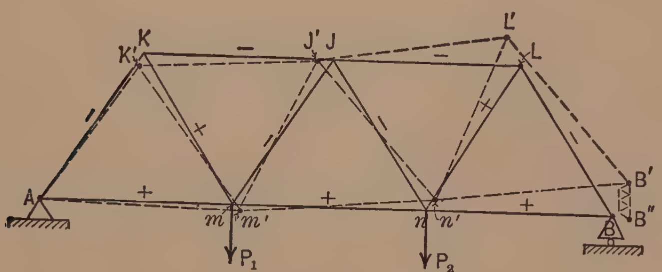


FIG. 35

As a simple illustration of the construction, the diagram of Fig. 34*a* is drawn for the truss of Fig. 34 with data as shown in Table A.

26. The Mohr Correction Diagram.—In the preceding example the truss had one point, A, which remained fixed and one member, $A - a$, which maintained a fixed direction throughout the process of deformation. For all such cases the construction proceeds directly. Now, in general, every stable frame will have one point and one line fixed, but the latter does not necessarily coincide with any member. In Fig. 35, loaded as shown, the point A is fixed, as is also the direction of the line AB, but every bar in the frame changes its direction. The Williot diagram, since it consists essentially in the repeated application of the construction shown in Fig. 33*a*, that is, the location of a third point from two others of *known location*, fails for the case of Fig. 35 unless emended. This emendation takes the following form: We assume any member, as AK, to be fixed, and draw the Williot diagram. This construction obviously gives the correct displacement of all joints *with respect to AK* and it only remains to determine the true position of AK. The deformed

truss is shown (greatly exaggerated) by the dotted lines in Fig. 35. Since the deformed configuration of the truss is correct, evidently we only need to rotate it as a rigid body until the point B' takes the position B'', it being a condition of the problem that the line through the joints A and B maintains a fixed horizontal position. If, then, to the displacement of any joint as determined by the Williot diagram we add vectorially the rotational displacement just explained, the result will be the true displacement. The angular rotation of the frame is sufficiently exactly expressed as $\frac{B'B''}{AB}$. Since every line in the truss must turn through this angle, the rotational displacement of the point J, for instance, will be $B'B'' \times \frac{AJ}{AB}$ in a direction normal to AJ. It is assumed here that since the deflections are small we may use the length and direction of AJ as identical with the length and direction of AJ'. A similar equation may be used to determine the rotational displacement of any other joint.

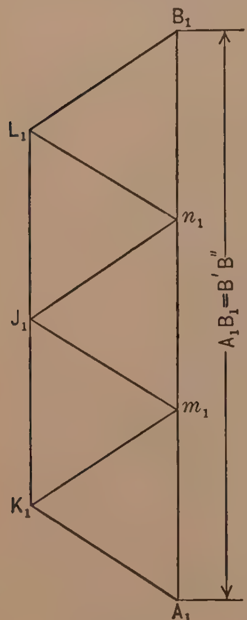


FIG. 35a

The graphical solution of the rotational displacement may be accomplished as follows: If upon the known displacement, $B'B''$, as a base, we construct a figure similar to the given frame (see Fig. 35a), turned through a right angle, since $B'B'' \perp AB$, we note that

$$(1)\ A_1m_1 \perp Am \quad \text{and} \quad (2)\ Am = A_1B_1 \cdot \frac{Am}{AB} = B'B'' \cdot \frac{Am}{AB};$$

likewise $L_1A_1 \perp AL$ and is equal to $B'B'' \cdot \frac{AL}{AB}$, and so on. These quantities must therefore correctly represent in magnitude and direction the desired rotational displacements.

This simple and elegant construction, without which the Williot diagram would be of limited usefulness, is known as Mohr's correction diagram.*

26a. Example.—Fig. 36 shows a Williot diagram for the truss of Fig. 35, drawn on the assumption that AK stands fast. The vectors Am', AK', AL', etc., represent the magnitudes and directions of the

* O. Mohr, "Ueber Geschwindigkeitspläne und Beschleunigungspläne," Zivilingenieur, 1887.

displacements of m, K, L, etc., referred to the point A and the line AK as fixed. As shown in the displacement diagram and in Fig. 35, this results in B lifting from the support an amount $B'B''$. To place the truss in its true position it is necessary to give it a rigid-body rotation

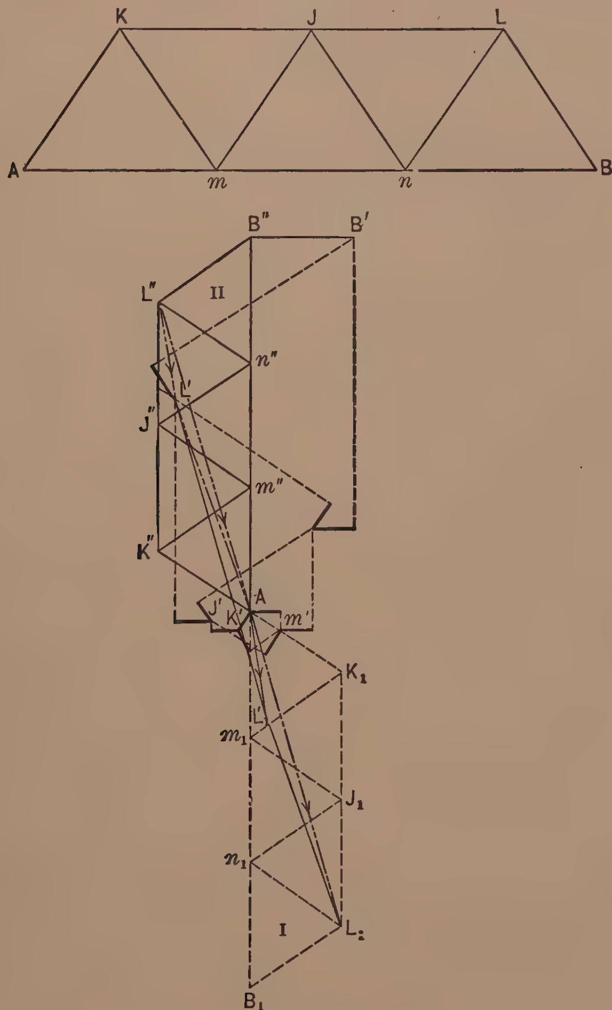


FIG. 36

about A until B lies in the same horizontal line with A. The Mohr correction diagram described above gives all the displacements due to this rotation, and is shown as I in the lower part of Fig. 36. AB_1 is laid off equal to the vertical component of AB' ($= B'B''$ of Fig. 35),

and a similar figure to the original truss, rotated through 90°, is drawn on this base. Then the rotational displacement of, say, point L, is AL_1. If we add the vectors AL' and AL_1 we get the final correct displacement AL'_1. However, a simpler and more compact diagram results if we draw the correction diagram as shown in the upper part of the figure as II. Here the rotational displacement of any point, as L, is $L''A$. This combined with AL' gives $L''L'$ as the final displacement. This is clearly identical with AL'_1. Similarly, the resultant deflection of n is $n''n'$, of J, $J''J'$, etc. Owing to the greater compactness, the correction diagram is always applied in this manner.

SECTION III.—SUMMARY AND APPLICATIONS

27. Recapitulation.—A brief recapitulation of the several methods for finding deflections may be of some aid to the student.

A. The method which was first developed and which is adopted as the standard method for this treatise is based on the equivalence of the external and internal work of a "dummy" *unit loading* (force or couple imagined to act at a point whose deflection is desired) *acting through displacement due to other causes*. It may thus be viewed as a special case of the general theorem of "Virtual Work," * though the derivation here given does not make explicit use of that principle.

In its application to the deflection of structures, the above method appears to have been first perceived by Maxwell (1864), but it was independently discovered by Mohr (1874) and its application greatly broadened. For brevity we shall refer to it as the "Maxwell-Mohr" method.† In the form here presented it is applicable to finding the displacement, linear or angular, of any point in a bar, or an assemblage of bars, straight or slightly curved, due to a distortion (taking place in any portion of any or all bars) which may be represented by a combination of axial and flexural deformation. In the cases we shall study, these deformations are generally due to applied loads, but it is most

* A statement of this principle as applied to rigid bodies may be found in almost any treatise on analytical mechanics, e.g., Church, p. 67 et seq. The deflection equation for a true framework follows easily from this, but the justification of the more general principle of Virtual Work as applied to deformable solids is by no means so simple and it is believed that the proof presented in the text, though less comprehensive, will present less difficulty to the student.

† The propriety of linking the names of Maxwell and Mohr with the method of the dummy unit loading is open to some question, since this particular method did not originate with either. (See Prof. I. P. Church, Trans. A. S. C. E., Vol. XXXIII, p. 649.) To them is due the general method of obtaining deflections by the principle of work, which is now universally applied by means of an arbitrary unit loading.

important for the student to note that this is not necessarily so. For example, if a member of a truss is shortened or lengthened by change of temperature, play in the pinholes or tightening of a turnbuckle, or if a beam has its temperature so varied that the fibers on one side are shortened and those on the other side lengthened in a manner similar to flexural distortion, the method will apply equally well. It is, of course, necessary that these changes shall be small to the order of elastic deformations. The distortions being known, the procedure is invariable: We apply a unit loading to the point where we wish the deflection, determine the moment m and the axial stress n (the shear if desired) for all sections of all members of the structure, and we have

$$\delta = (\text{numerically}) \Sigma \int n \cdot \Delta ds + \Sigma \int m \cdot \Delta d\phi,$$

and

$$\alpha = (\text{numerically}) \Sigma \int n_\alpha \cdot \Delta ds + \Sigma \int m_\alpha \cdot \Delta d\alpha.$$

B. The method based on the derivatives of internal work, "Castigliano's theorem," differs in fundamental conception from the Maxwell-Mohr method, but in the application to the deflection of structures the scope of the two methods is virtually the same. We have seen that the "m" and "n" of the Maxwell-Mohr equations are identical with $\frac{\partial M}{\partial P}$ and $\frac{\partial N}{\partial P}$ of Castigliano's equations. The latter can be extended readily to include temperature changes, yielding supports, etc., and the essential difference in the methods lies in the way in which, say, m is obtained in the one method and $\frac{\partial M}{\partial P}$ in the other.

In the former case we apply a unit loading at the point we are investigating and write the expression for m from the rules of statics; in the latter we set up the expression for M due to the specified loading (including, if necessary, a load P at the point of deflection) and differentiate this with respect to P, giving P its numerical value in M and $\frac{\partial M}{\partial P}$ *after* the operation. Recalling that M and N are linear functions of the loads, i.e.,

$$M = C_1P_1 + C_2P_2 \dots + C_rP_r \dots + C_nP_n,$$

we see that

$$\frac{\partial M}{\partial P_r} = C_r \; (= m_r \text{ obviously}),$$

and the operation is thus simpler than might at first appear.

C. The method of moment areas affords a very simple treatment of

angular and linear deflection of beams based upon the two propositions (*a*) that the relative tangential rotation due to flexure between any two points of a beam is equal numerically to the area of the $\frac{M}{EI}$ diagram between the two points, and (*b*) that the deflection of any point of a bent beam referred to a tangent at some other point is numerically equal to the statical moment of the $\frac{M}{EI}$ diagram lying between the points, about normal through the first point. These principles, though deducible as corollaries of the work theorem, are easily deduced from the most elementary considerations in the geometry of the strained beam. Their application to problems is clear-cut and direct and requires no comment. This method is applicable to all beam deflection problems, and to trusses which act approximately as beams. Where the areas and moments of areas are not readily handled algebraically, useful approximations are easily made by taking a finite summation of reasonably small elements of area (and their moments). The method is not applicable to truss deflection problems except as above noted.

D. The method of elastic weights is here used to include all the methods having for their basis the correspondence between the deflection curve of a structure and the moment diagram of a beam subjected to an imagined set of "elastic loads." The fundamentals of the method are quite fully set forth in Section II C, where it is shown that the same general method is directly applicable to the vertical deflections of both beams and trusses. It may also be extended to obtain the horizontal deflections of trusses * and is therefore a method capable of very wide application. So far as the application to beam deflections is concerned, it is an alternative and strictly parallel method to that of moment areas.

From its basic character in treating the deflection diagram as a moment curve for a properly adjusted fictitious loading, it lends itself directly to both graphical and analytical calculation, and to approximate calculation as noted in preceding paragraph on moment areas.

E. The Williot diagram affords a direct method, based on purely geometrical considerations, for obtaining the actual (as distinct from the component) displacements in any true truss, to which alone it is applicable.

Each of the above methods is independent; that is, each may be deduced without the aid of the other.

28. Comparative Advantages of Methods.—Some remarks on comparative advantages have already appeared in the preceding pages, and

* See paper by Professor W. S. Kinne, "Wisconsin Engineer," Feb., 1920.

some further discussion may be found in Chapters II and III. The following points will bear emphasis here.

(*a*) If it is desired to find the angular or linear displacement of a simple or cantilever beam at a single section, calculation by moment areas or elastic weights will nearly always prove the most expeditious method.

(*b*) If the simultaneous deflection of a number of points is wanted, the construction of the elastic curve as a string polygon (see page 54) is recommended as the most advantageous method.

In either of the above cases, if (from tables or otherwise) the general equation of the elastic curve is known, a simple substitution gives any deflection, and of course this will be the easiest solution. However, complete solutions of the equation $\frac{d^2y}{dx^2} = \frac{M}{EI}$ are not usually available in advance for any but the simplest cases, and the integration of the

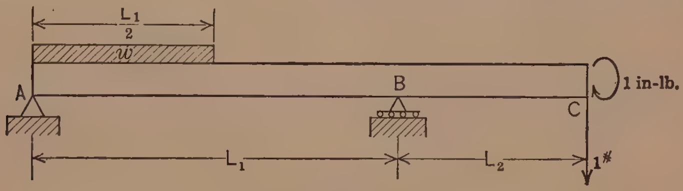


FIG. 37

equation and the determination of the constants is in general a far more difficult method of solution than those suggested above.

(*c*) To obtain the displacement of a single point in a truss, the equation $\delta = \sum \frac{SuL}{AE}$ will usually give the readiest solution.

(*d*) To obtain the simultaneous displacements of a number of points in a truss, the Williot diagram is the simplest and quickest method. We may also repeat the calculation of (*c*) for each point, or we may apply the method of elastic weights as described in Art. 20 and illustrated in problem II, page 78. The latter method is the quicker of the two and for practical purposes is equally exact. The Williot diagram is open to the same criticism as the ordinary stress diagram and many other graphical processes; small errors easily creep in and may become cumulative and so introduce important error in the final result. With reasonable care in construction, however, the Williot diagram will probably give results as accurate as the data justify. It will be largely used for truss deflection problems in the later chapters of this book.

29. Examples.

Problem I (Fig. 37).—To find linear and angular displacement at *C*.

(*a*) By the Maxwell-Mohr method (Dummy unit loading). With origin at A we have

$$M = \begin{cases} \tfrac{3}{8}wL_1x - \dfrac{wx^2}{2} \ldots A \text{ to } \dfrac{L_1}{2}, \\ \tfrac{3}{8}wL_1x - \dfrac{wL_1}{2}\left(x - \dfrac{L_1}{4}\right) \ldots \dfrac{L_1}{2} \text{ to } B. \end{cases}$$

From B to C, $M = O$, hence $\displaystyle\int \frac{Mmdx}{EI}$ vanishes for this section.

Since we are concerned with vertical deflection, we apply the unit load at C downwards; the sense is a matter of indifference so long as due regard is paid to the sign of m. We have

$$m = -x\frac{L_2}{L_1} \text{ from } A \text{ to } B,$$

whence, assuming E and I constant,

$$\delta_C = \int_A^B \frac{Mmdx}{EI} = \frac{1}{EI}\left\{\int_0^{\frac{L_1}{2}}\left(\tfrac{3}{8}wL_1x - \frac{wx^2}{2}\right)\left(-x\frac{L_2}{L_1}\right)dx\right.$$

$$\left. + \int_{\frac{L_1}{2}}^{L}\left[\tfrac{3}{8}wL_1x - \frac{wL_1}{2}\left(x - \frac{L_1}{4}\right)\right]\left(-x\frac{L_2}{L_1}\right)dx\right\}$$

$$= \frac{1}{EI}\left\{-\int_0^{\frac{L_1}{2}}\tfrac{3}{8}wL_2x^2dx + \int_0^{\frac{L_1}{2}}\frac{wL_2}{2L_1}x^3dx + \int_{\frac{L_1}{2}}^{L_1}\tfrac{1}{8}wL_2x^2dx\right.$$

$$\left. - \int_{\frac{L_1}{2}}^{L_1}\frac{wL_1L_2xdx}{8}\right\}$$

$$= \frac{w}{EI}\left\{-\left[\tfrac{1}{8}L_2x^3\right]_0^{\frac{L_1}{2}} + \left[\frac{L_2}{8L_1}x^4\right]_0^{\frac{L_1}{2}} + \left[\frac{L_2x^3}{24}\right]_{\frac{L_1}{2}}^{L_1} - \left[\frac{L_1L_2}{16}x^2\right]_{\frac{L_1}{2}}^{L_1}\right\}$$

$$= \frac{wL_1^3L_2}{EI}\left[-\frac{1}{64} + \frac{1}{128} + \frac{7}{192} - \frac{3}{64}\right] = -\frac{7}{384}\frac{wL_1^3L_2}{EI},$$

i.e., the displacement is *upward* by this amount.
Also,

$$\alpha_C = \int_A^B \frac{Mm_\alpha dx}{EI},$$

where m_α = moment at any section due to a unit couple at C acting as shown. This couple will cause a negative reaction of $\frac{1\#}{L_1}$ at A, whence,

$$m_\alpha = -\frac{x}{L_1}, \text{ from } A \text{ to } B.$$

It is evident then that the detail work is exactly as above with $-\frac{x}{L_1}$ substituted for $-x\frac{L_2}{L_1}$, whence

$$\alpha_C = -\frac{7}{384}\cdot\frac{wL_1^3}{EI},$$

the minus sign indicating a rotation opposite to that shown, i.e., a counter-clockwise rotation.

Since the beam from B to C is unstressed, it is clear (α and δ being very small quantities) that

$$\delta_C = L_2 \cdot \alpha_C.$$

It is also evident that $\alpha_C = \alpha_B$. As a check we may compute α_A. Applying a unit couple clockwise at A and taking origin at B, we have

$$\alpha_A = \int_A^B \frac{Mmdx}{EI}$$

$$= \frac{1}{EI}\left\{\int_0^{\frac{L_1}{2}} \frac{wL_1x}{8}\cdot\frac{x}{L_1}dx + \int_{\frac{L_1}{2}}^{L_1}\left[\frac{wL_1x}{8} - \frac{w\left(x-\frac{L_1}{2}\right)^2}{2}\right]\frac{x}{L_1}\right\} = +\frac{9}{384}\,\frac{wL_1^3}{EI},$$

i.e., the rotation at A is clockwise. It is evident from symmetry that if the beam is fully loaded,

$$\alpha_A = \frac{9}{384}\,\frac{wL_1^3}{EI} + \frac{7}{384}\,\frac{wL_1^3}{EI} = \frac{1}{24}\,\frac{wL_1^3}{EI}.$$

This is a well-known result easily verified by the general method. Thus for full loading

$$\alpha = \int_A^B \frac{Mmdx}{EI} = \frac{1}{EI}\int_0^{L_1}\left(wL_1x - \frac{wx^2}{2}\right)\frac{x}{L_1}dx$$

$$= \frac{1}{EI}\left[\frac{wx^3}{6} - \frac{wx^4}{8L_1}\right]_0^{L_1} = \frac{1}{24}\,\frac{wL_1^3}{EI}, \textit{ check.}$$

(*b*) By Castigliano's theorem of the partial derivative of the work of deformation.

Suppose an arbitrary load P to act downward at C. Then, with origin at A, and calling moment due to w, M_w, we have

$$M = \begin{cases} M_w - Px\dfrac{L_2}{L_1},\ A \text{ to } B \\ -P(L_1 + L_2 - x),\ B \text{ to } C' \end{cases} \qquad \frac{\partial M}{\partial P} = \begin{cases} -x\dfrac{L_2}{L_1},\ A \text{ to } B \\ -(L_2 + L_1 - x),\ B \text{ to } C \end{cases}$$

$$\therefore\ \delta_C = \int_A^C \frac{Mdx}{EI}\cdot\frac{\partial M}{\partial P} = \int_P^B \frac{\left(M_w - Px\dfrac{L_1}{L_1}\right)dx}{EI}\cdot\left(-x\frac{L_2}{L_1}\right)$$
$$+\int_B^C \frac{P(L_2 + L_1 - x)dx}{EI}(L_2 + L_2 - x).$$

Since this holds for all values (not infinite) of P, it will be true if we assume $P = O$. Then

$$\delta_C = \int_A^B \frac{M_w dx}{EI}\left(-x\frac{L_2}{L_1}\right),$$

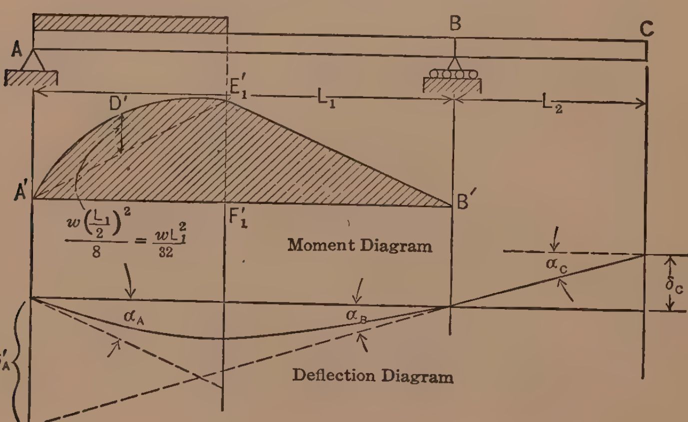


Fig. 38

which is identical with the corresponding equation in (a); hence we need not carry the detail further.

(c_1) By the method of Elastic Weights.

Since $\delta_C = \alpha_C \cdot L_2$, and $\alpha_C = \alpha_B$, the problem is practically solved when α_B is found.

The rotation at B is numerically equal to the shear at B in the beam AB when the $\dfrac{M}{EI}$ diagram is applied as a load curve. Since we are assuming E and I constant it will be convenient to work for $EI\alpha_B$. Fig. 38 shows the moment diagram for the given loading. The moment area may be divided as indicated into the triangle $A'E_1'B'$ and the

parabola $A'D'E_1'$. Since the latter is identical with the moment diagram for a simple beam span equal to AE, its area

$$= \frac{2}{3}\cdot\frac{L_1}{2}\cdot\frac{wL_1^2}{32} = \frac{wL_1^3}{96}.$$

The area of the triangle

$$= \frac{wL_1}{2}\cdot\frac{1}{4}\cdot\frac{L_1}{2}\cdot\frac{L_1}{2} = \frac{wL_1^3}{32}.$$

Therefore $EI\alpha_B$ = shear at B due to M-diagram applied to AB

$$= -\frac{\text{Mom. of } A'E_1'B' + \text{Mom. } A'D'E_1'}{L_1}$$

$$= -\left[\frac{wL_1^3}{32}\cdot\frac{1}{2} + \frac{wL_1^3}{96}\cdot\frac{1}{4}\right] = -\frac{7}{384}wL_1^3.$$

(c_2) By method of Moment Areas denoting clockwise rotation as positive, it is evident that

$$\alpha_C = \alpha_B = -\frac{\delta'_A}{L_1};$$

and δ'_A = Moment of $A'E_1'B'$ about A + moment of $A'D'E_1'$ about A

$$= \frac{wL_1^3}{32}\cdot\frac{L_1}{2} + \frac{wL_1^3}{96}\cdot\frac{L_1}{4} = \frac{7}{384}wL_1^4,$$

$$\therefore\ \alpha_B = -\frac{7}{384}wL_1^3$$

Problem II.—Given the truss of Fig. 39*a* to find the vertical deflections of the lower chord joints.

(*a*) By the Maxwell-Mohr method.

Table A, Fig. 39*c* shows the detail of the work and the results. To avoid repeated division by E with the resulting small decimals, it is simpler to work first for $E\delta = \sum\frac{SuL}{A}$. For convenience in tabulating, $\frac{SL}{A}$ is taken in units of $\frac{1000\#}{\text{ins.}}$, and consequently the dummy unit load is 1000#. The stresses S are obtained from the Maxwell diagram of Fig. 39*b*. For the particular loading of this problem it is evident that $u_d = \frac{S}{P}$. For most cases, of course, no such simple relation exists and u_d would be obtained from an independent diagram or by analytical computation. It is clear from the symmetry of the truss that u_b is obtained directly from u_d, and that u_c will be the same on either side of the center and will

equal 2 × (corresponding value of u_d for left half of truss). The sign of the quantity $\frac{SuL}{A}$ will obviously be positive when u and S have the same sign; otherwise it will be negative. The remainder of the calcula-

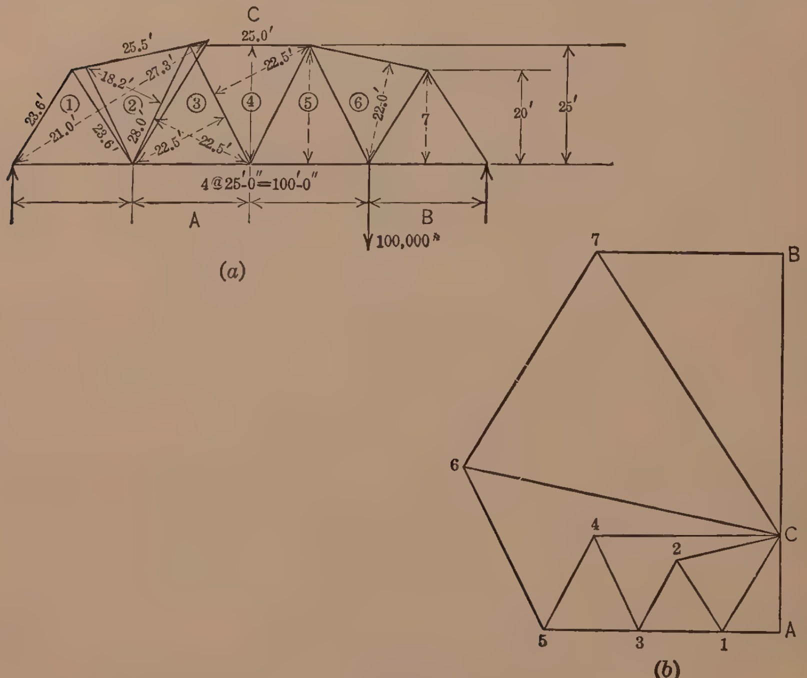


FIG. 39

tion requires no explanation and the resulting deflections are given at the bottom of Table A.

(b) By Castigliano's method.

The fundamental equation is

$$\delta_r = \frac{\partial W}{\partial P_r} = \frac{\partial}{\partial P_r}\left(\frac{1}{2}\sum\frac{S^2L}{AE}\right) = \sum\frac{SL}{AE}\cdot\frac{\partial S}{\partial P_r}.$$

Recalling that

$$S = k_1P_1 + k_2P_2 \ldots + k_rP_r \ldots + k_nP_n,$$

we readily obtain

$$\frac{\partial S}{\partial P_r} = k_r.$$

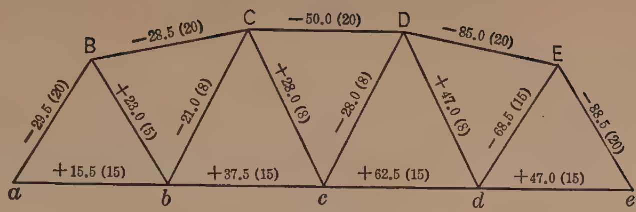


FIG. 39c

NOTE—
The figures appearing first are the stresses, S.
The figures in parenthesis are the sectional areas.

TABLE A

1	2	3	4	5	6	7		8	9		10
Member	A, Sq. In.	L, In.	S, Lbs.	$\frac{SL}{A}$ $\frac{1000\#}{\text{In}^2}$	u_b, Lbs. 1000# at b	$\frac{SL}{A}\cdot u_b$ +	−	u_c 1000# at c	$\frac{SL}{A}\cdot u_c$ +	−	$\frac{SL}{A}\cdot u_d$
aB	20	283	−29,500	− 416.0	−885	368,000	....	− 590	246,000	....	123,000
BC	20	306	−28,500	− 436.0	−850	371,000	....	− 570	248,000	....	124,000
CD	20	300	−50,000	− 750.0	−500	375,000	....	− 1000	750,000	....	375,000
DE	20	306	−85,000	−1300.0	−285	371,000	....	− 570	740,000	....	1,110,000
Ee	20	283	−88,500	−1255.0	−295	371,000	....	− 590	740,000	....	1,120,000
ab	15	300	+15,500	+ 310.0	+470	146,000	....	+ 310	96,000	....	46,500
bc	15	300	+37,500	+ 750.0	+625	469,000	....	+ 750	564,000	....	281,000
cd	15	300	+62,500	+1250.0	+375	469,000	....	+ 750	940,000	....	780,000
de	15	300	+47,000	+ 940.0	+155	146,000	....	+ 310	292,000	....	442,000
Bb	15	283	+23,000	+ 434.0	+685	297,000	....	+ 460	199,000	....	100,000
bC	8	336	−21,000	− 884.0	+470	....	416,000	− 420	....	372,000	186,000
Cc	8	336	+28,000	+1260.0	−280	....	355,000	+ 560	....	705,000	353,000
cD	8	336	−28,000	−1260.0	+280	....	355,000	+ 560	705,000	....	353,000
Dd	8	336	+47,000	+2115.0	−210	....	444,000	− 420	895,000	....	990,000
dE	15	283	+68,500	+1290.0	+230	297,000	....	+ 460	595,000	....	885,000

$E = 28{,}000{,}000$ $\qquad E \cdot \delta_b = 2{,}110{,}000$, $\delta_b = .074$ in. $\qquad E \cdot \delta_c = 4{,}887{,}000$, $\delta_c = .174$ in. $\qquad E\delta_d = 7{,}268{,}000$, $\delta_d = .258$ in.

But, S(due to P_r) $= P_r \cdot u_r$, i.e. $u_r = k_r = \dfrac{\partial S}{\partial P_r}$, hence it is evident that the detail of the solution by Castigliano's method reduces to the same form as for the Maxwell-Mohr method.

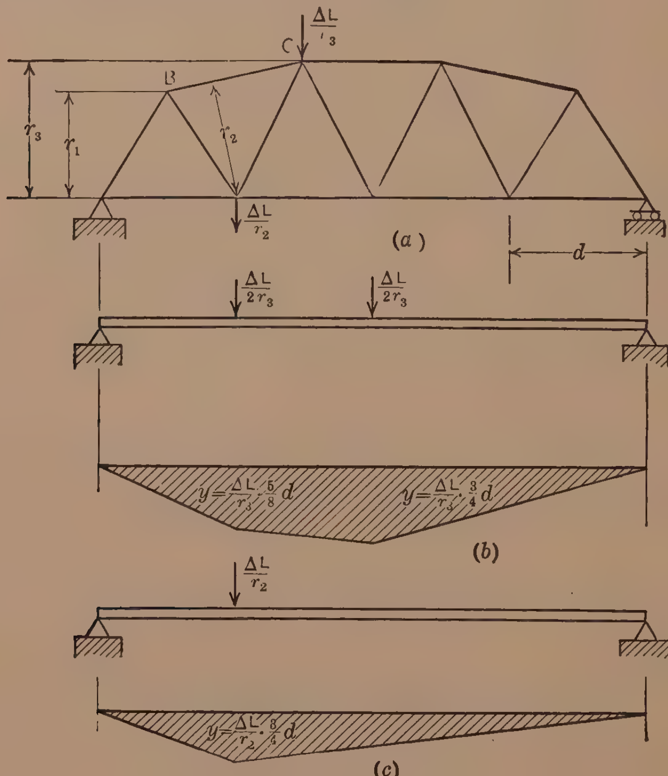


FIG. 40

GENERAL NOTES

Fundamental Equation:—

$\delta_{n-r} = \Delta Lr \times u_{n-r}$, where δ_{n-r} = deflection of joint "n" due to a change of length ΔLr in member "r"; and u_{n-r} = stress in "r" due to unity at n.

Fundamental Working Rule:—

Let an "elastic load" for each member (= change of length ÷ moment arm) be applied to truss *at the moment center corresponding to the member. The simple beam moment diagram* for this fictitious loading is the actual deflection diagram for the truss joints.

Deflections due to Chord Distortions:—

Fig. 40 (c) shows the method of application of elastic loads for a deformation ΔL in BC. For any other upper chord member the method is identical. The same essential procedure is followed for lower chord distortions if deflections of both upper and lower chord joints are desired. If displacement diagram for lower chord joints only is wanted, the procedure is shown (for the chord member bc) in Fig. 40 (b).

(*c*) By the method of Elastic Weights.

We proceed as in (*a*) to find $E \cdot \Delta L = \dfrac{SL}{A}$. Corresponding to each member an "elastic load" $= \dfrac{E\Delta L}{r}$ is to be applied to the truss at the moment center for the given member. r is the "arm" of the member referred to its moment-center. In the case of web members, an equivalent substitute loading is generally used in place of $\dfrac{E \cdot \Delta L}{r}$.

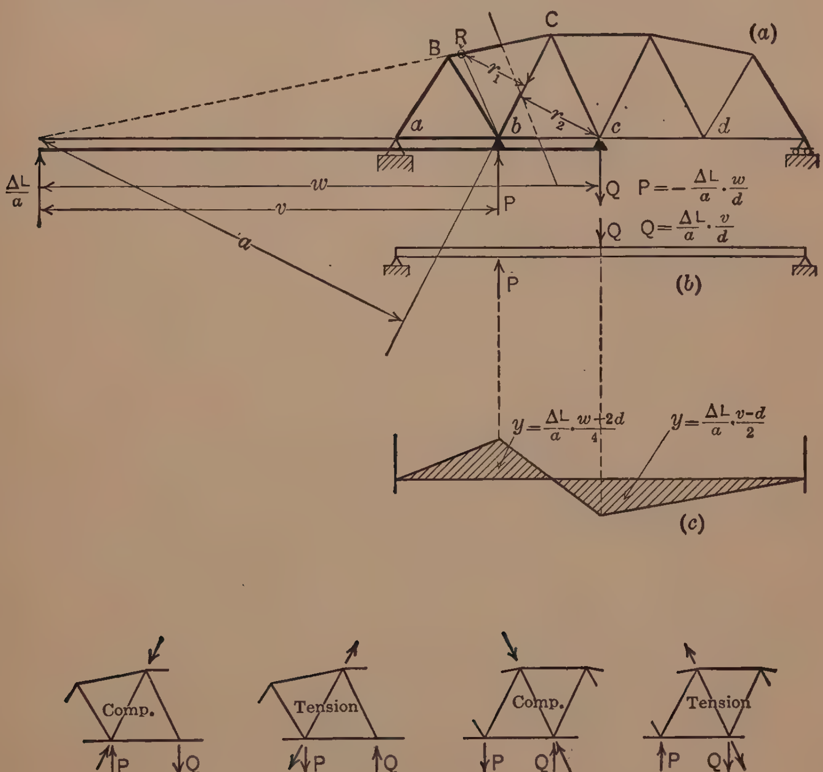


FIG. 41

NOTE

From similar triangles we have $\dfrac{r_1}{a} = \dfrac{d}{w}$ and $\dfrac{r_2}{a} = \dfrac{d}{v}$, $(Rb \| Cc)$, whence $P = -\dfrac{\Delta L}{r_1}$, $Q = \dfrac{\Delta L}{r_2}$. This rule is general.

Elastic loads for chords act in direction of actual loads. The manner of application of P and Q corresponding to tension and compression in webs is illustrated in diagrams at bottom of Fig. 41.

P is always the nearer load to the *moment center*.

TABLE B

Chord Members *					Web Members						
Joint	Member	$E \times \Delta L$	r	$\frac{E \times \Delta L}{r}$	Joint	Member	$E \times \Delta L$	r: r_1	r: r_2	$P = \frac{E\Delta L}{r_1}$	$Q = \frac{E\Delta L}{r_2}$
B	ab	310,000	20	+15,500	b	Bb	+ 434,000		27.3		+ 15,900
C	bc	750,000	25	+ 30,000	b	Cb	− 884,000	18.2		− 48,400	
D	cd	1,250,000	25	+ 50,000	c				22.5		+ 39,300
E	de	940,000	20	+ 47,000	b	Cc	+1,260,000	22.5		− 56,000	
b	aB	416,000	21	+ 19,800	c				22.5		+ 56,000
	BC	436,000	22	+ 19,800	c	cD	− 1,260,000		22.5		− 56,000
c	CD	750,000	25	+ 30,000	d			22.5		+ 56,000	
d	DE	1,300,000	22	+ 59,000	c	dD	+2,115,000		22.5		− 93,800
	Ee	1,255,000	21	+ 59,800	d			18.2		+1,160,000	
					d	dE	+1,290,000		27.3		+ 47,400

* End post is treated as a chord.

This is explained in Fig. 41. The values of r used are tabulated on the figure in 39a. The remainder of the process will be clear from Figs. 40, 41 and 42, and Table B. When the resultant values of the elastic loads have been obtained the moments may be calculated analytically, or the elastic load moment curve = true deflection curve may be constructed as a string polygon (Fig. 42).

(d) By the Williot-Mohr diagram.

With the data of column 5, Table A (Fig. 39c), assuming member aB to stand fast, the Williot diagram of Fig. 43 is constructed after the method explained in section II-e. The Mohr correction diagram is then applied as was explained in the illustrative problem in this section (Fig. 36). The deflection results for b, c, and d are indicated in the figure.

The very close check obtained by the three independent methods is worthy of note.

Problem III.—Figs. 44a and b. This is a beam deflection problem similar to I. It is solved by the method of elastic weights, and the detail work is fully shown.

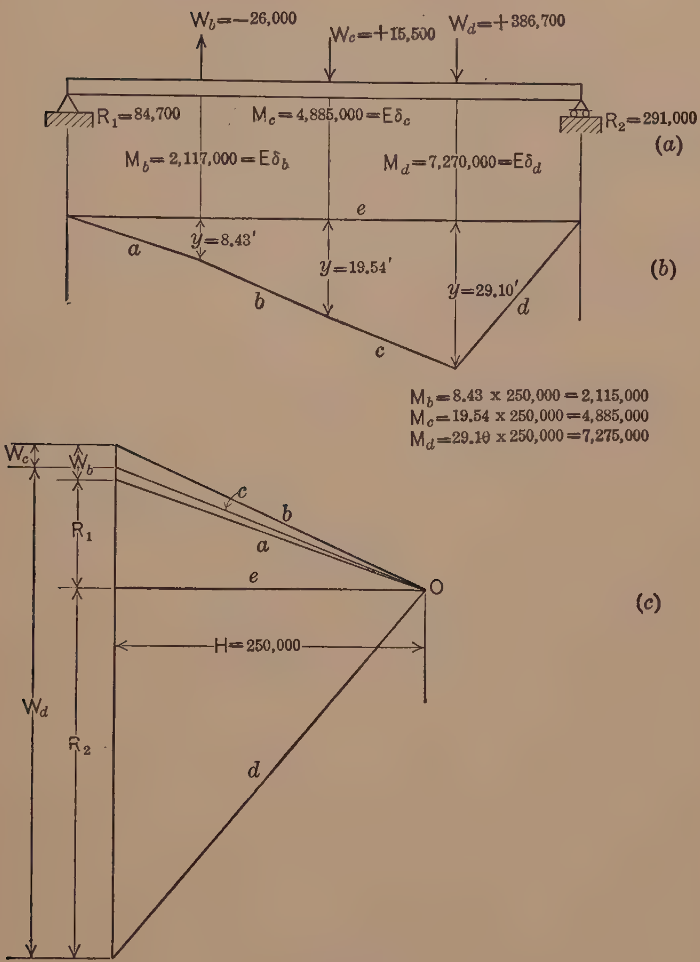


FIG. 42

PROBLEM III

REACTIONS AND MOMENTS FOR ACTUAL LOADING

$$R_L = \frac{3000 \times 6 + 1800 \times 13.5}{18} = 2350.$$

$$R_R = \qquad\qquad 2450.$$

$$M_B = 2350 \times 9 - \frac{200 \times 9^2}{2} = 13{,}050'^{\#} = 156{,}600''^{\#}.$$

$$M_C = 2450 \times 8 - 3000 \times 2 = 13{,}600'^{\#} = 163{,}200''^{\#}.$$

$$M_D = 2450 \times 6 = 14{,}700'^{\#} = 176{,}400''^{\#}.$$

MOMENTS AND SHEARS FOR BEAM LOADED WITH MOMENT DIAGRAM

		Area	× Arm (E)	= Moment
Area I	$= \frac{2}{3} \times 24{,}300 \times 9$ =	145,800	13.5	1,970,000
II	$= \frac{1}{2} \times 156{,}600 \times 9$ =	705,000	12	8,460,000
III	$= 156{,}600 \times 3$ =	469,800	7.5	3,525,000
IV	$= 19{,}800 \times \frac{1}{2} \times 3$ =	29,700	7	207,900
V	$= 176{,}400 \times \frac{1}{2} \times 6$ =	529,200	4	2,116,800
		1,879,500		16,279,700

$$R_L = \frac{16{,}279{,}700 \times 144}{18 \times 12} = 905{,}000 \times 12$$

$$\begin{aligned}
\text{Area III}_1 &= 156{,}600 \\
\text{Area IV}_1 &= 6{,}600 \times .5 \times 1 = 3{,}300 \\
\text{Shear at } C &= 905{,}000 - 145{,}800 - 705{,}000 - 156{,}600 - 3{,}300 \\
&= 105{,}700 \times 12
\end{aligned}$$

$$\alpha_C = \frac{-105{,}700 \times 12}{30{,}000{,}000 \times 144} = -.000{,}294 \text{ Radians} = \text{Slope of Beam at } C.$$

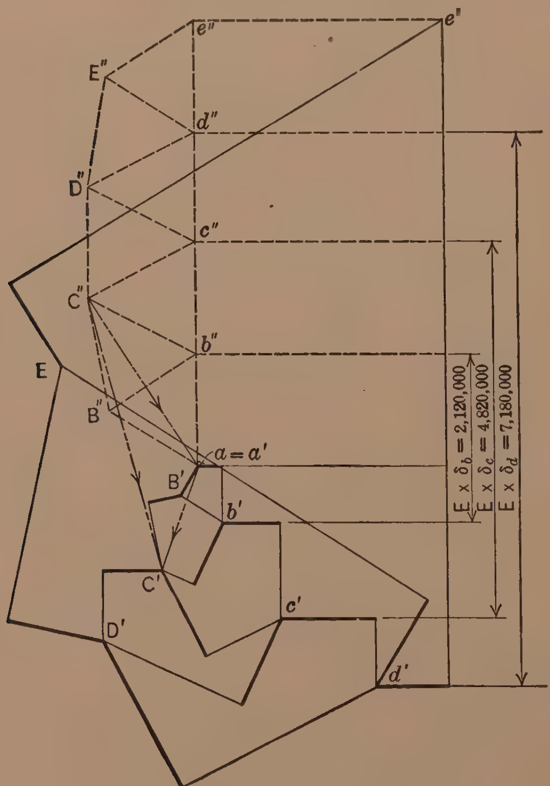


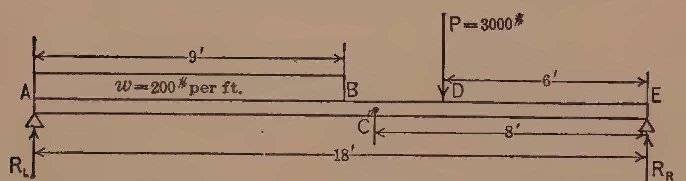
FIG. 43

Bending Moment at C

$$\begin{array}{lrlrrr} R_L & = & 905{,}000 \times 10 & = & + & 9{,}050{,}000 \\ \text{I} & = - & 145{,}800 \times 5.5 & = & - & 802{,}000 \\ \text{II} & = - & 705{,}000 \times 4 & = & - & 2{,}820{,}000 \\ \text{III} & = - & 156{,}000 \times .5 & = & - & 78{,}300 \\ \text{IV} & = - & 3{,}300 \times .33 & = & - & 1{,}100 \end{array}$$

$$M_c = -5{,}384{,}100 \times 144$$

$$\delta_C = \frac{5{,}384{,}100 \times 144}{30{,}000{,}000 \times 144} = .1782'' = \text{Deflection at } C.$$

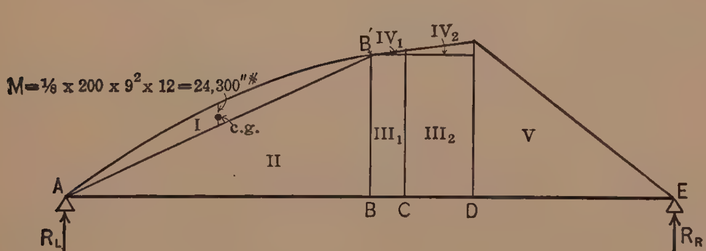


$E = 30{,}000{,}000$#/sq. in.

$I = 144$ ins.^{4}

Required α_c and δ_c (vertical)

(a)

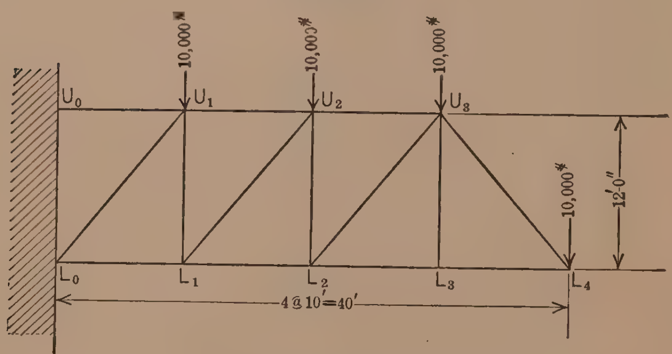


(b)

FIG. 44

Problems IV, V and VI, Figs. 45, 46 and 47 are problems in truss deflections. The accompanying tables show the detail work.

PROBLEM IV



$E = 28{,}000{,}000^{\#}/\text{sq. in.}$
Required: Vertical Deflection at L_2

FIG. 45.

Member	Length (L) Inches	Stress (S) Pounds	Area (A) Sq. In.	$\frac{SL}{EA}$	u	$\frac{SuL}{EA}$
U_0U_1	120	+83,400	6.00	+.0595	+1.665	+.0990
U_1U_2	120	+50,000	6.00	+.0357	+ .832	+.0297
U_2U_3	120	+25,000	6.00	+.0178	0	0
L_0L_1	120	−50,000	6.00	−.0357	− .832	+.0297
L_1L_2	120	−25,000	5.00	−.0214	0	0
L_2L_3	120	− 8,330	4.00	−.0089	0	0
L_3L_4	120	− 8,330	3.00	−.0119	0	0
U_1L_1	144	+30,000	5.00	+.0308	+1.000	+.0308
U_2L_2	144	+20,000	4.00	+.0257	+1.000	+.0257
U_3L_3	144	0	3.00	0	0	0
U_3L_4	187	+13,000	4.00	+.0217	0	0
U_3L_2	187	−26,000	4.00	−.0434	0	0
U_2L_1	187	−39,000	4.00	−.0651	−1.300	+.0846
U_1L_0	187	−52,000	4.00	−.0868	−1.300	+.1128

Total deflection of L_2 = +.4123″

PROBLEM V

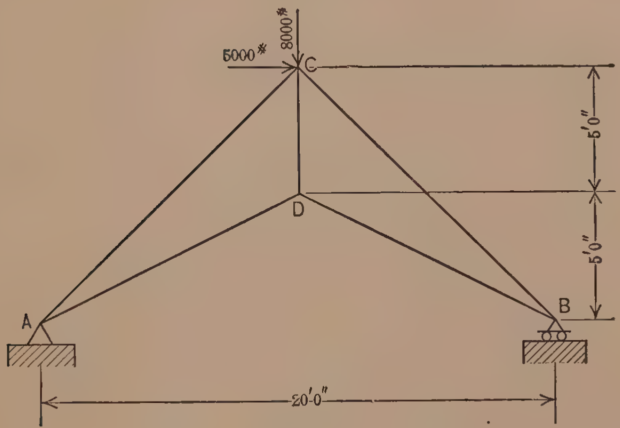


FIG. 46.

(1) HORIZONTAL DEFLECTION OF B

Member	Length (L) Inches	Stress (S) Pounds	Area (A) Sq. In.	$\frac{SL}{AE}$	u	$\frac{SuL}{AE}$
AC	170	−11,330	3.00	−.0229	−1.416	+.0324
CB	170	−18,400	3.00	−.0382	−1.416	+.0541
AD	134	+14,500	2.00	+.0347	+2.235	+.0775
DB	134	+14,500	2.00	+.0347	+2.235	+.0775
CD	60	+13,000	4.00	+.00696	+2.000	+.0139

Total horizontal deflection of B to right = .2554″

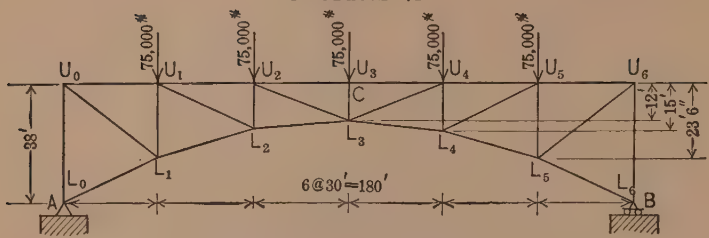
E is taken as 28,000,000* per sq. in.

(2) HORIZONTAL DEFLECTION OF C

Member	Length (L) Inches	Stress (S) Pounds	Area (A) Sq. In.	$\frac{SL}{EA}$	u	$\frac{SuL}{EA}$
AC	170	−11,330	3.00	−.0229	0	.0000
CB	170	−18,400	3.00	−.0382	−1.416	+.0541
AD	134	+14,500	2.00	+.0347	+1.117	+.0387
DB	134	+14,500	2.00	+.0347	+1.117	+.0387
CD	60	+13,000	4.00	+.00696	+.1000	+.00696

Total horizontal deflection of C to right = .13846″

PROBLEM VI



$E = 28{,}000{,}000^{\#}/\text{sq. in.}$

FIG. 47.

REQUIRED VERTICAL DEFLECTION AT C

Member	Length (L) Inches	Stress (S) Pounds	Area (A) Sq. In.	$\frac{SL}{EA}$	u	$\frac{SuL}{EA}$
U_0U_1	360	−239,000	10.00	− .307	− .6375	+ .1955
U_1U_2	360	−600,000	10.00	− .771	−2.000	+1.542
U_2U_3	360	−844,000	10.00	−1.085	−3.750	+4.070
L_0L_1	400	0	20.00	0	0	0
L_1L_2	374	+249,000	18.00	+ .1845	+ .661	+ .122
L_2L_3	362	+603,000	16.00	+ .4325	+ .2015	+ .088
U_0L_1	457	+304,000	10.00	+ .496	+ .810	+ .401
U_1L_2	402	+404,000	8.00	+ .725	+1.525	+1.105
U_2L_3	388	+274,500	6.00	+ .634	+1.880	+1.191
U_0L_0	456	−187,500	10.00	− .3055	− .500	+ .1578
U_1L_1	282	−255,000	8.00	− .3215	− .682	+ .219
U_2L_2	180	−170,500	6.00	− .1827	− .700	+ .128
U_3L_3	144	− 37,500*	2.00*	− .0964	− .500*	+ .0482

*One-half Actual Value. 9.2675″

The total vertical deflection of C is twice this result = 18.535″

VI$_b$.—The truss, loading and E are taken the same as in Fig. 47. Required: The horizontal movement of B.

Member	Length (L), In.	Stress (S), Lbs.	Area (A), Sq. In.	$\frac{SL}{EA}$	u	$\frac{SuL}{EA}$
U_0U_1	360	−239,000	10.00	− .307	− .617	+ .1892
U_1U_2	360	−600,000	10.00	− .771	−1.533	+1.182
U_2U_3	360	−844,000	10.00	−1.085	−2.165	+2.350
L_0L_1	400	0	20.00	0	+1.11	0
L_1L_2	374	+249,000	18.00	+ .1845	+1.678	+ .3095
L_2L_3	362	+603,000	16.00	+ .4325	+2.55	+1.102
U_0L_1	457	+304,000	10.00	+ .496	+ .785	+ .389
U_1L_2	402	+404,000	8.00	+ .725	+1.025	+ .744
U_2L_3	388	+274,500	6.00	+ .634	+ .680	+ .431
U_0L_0	456	−187,500	10.00	− .3055	− .483	+ .1473
U_1L_1	282	−255,000	8.00	− .3215	− .458	+ .1473
U_2L_2	180	−170,500	6.00	.1827	− .253	+ .0462
U_3L_3	144	− 37,500	2.00	.0964	.000	.0000

The total horizontal movement of B is twice this result = 14.075″ 7.0375″

PROBLEM VII

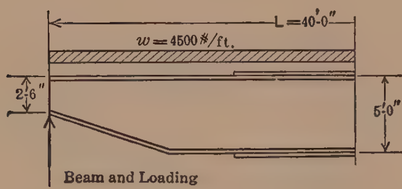


Fig. 48

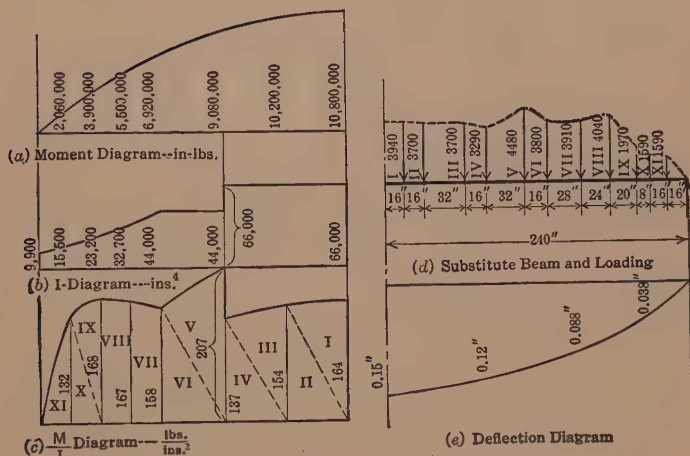


Fig. 49.

Calculation of Center Deflection

$$\delta_c = \frac{\overline{M}_c}{E}, \text{ if } \overline{M}_c = \text{center moment due to } \frac{M}{I} \text{ loading}$$

3940×16	=	63,000
3700×32	=	118,500
3700×64	=	236,300
3290×80	=	263,000
4480×112	=	503,000
3800×128	=	487,000
3910×156	=	610,000
4040×180	=	728,000
1970×200	=	394,000
1590×208	=	331,000
1590×224	=	356,000
		4,080,800 lbs./ins.

$R = 36{,}000$ lbs./ins.;

$$\overline{M}_c = R \times 240 - \Sigma Pa$$
$$= 36{,}000 \times 240 - 4{,}080{,}000$$
$$= 4{,}540{,}000$$
$$\delta_c = \frac{\overline{M}_c}{E} = \frac{4{,}540{,}000 \text{ lbs./ins.}}{30{,}000{,}000 \text{ lbs./ins.}^2}$$
$$= .1508 \text{ ins.}$$

Problem VII.—Fig. 48 is an example of the calculation of beam deflections where I is not constant. The method of elastic weights is used advantageously here. The $\frac{M}{I}$ diagram is plotted (Fig. 49*c*) and this is then applied as a load curve to a simple beam of same span as given beam, but of constant section (Fig. 49*d*). The moment diagram for this substitute beam and loading is the true deflection curve (Fig. 49*e*). In Fig. 49*c*, the $\frac{M}{I}$ areas I, II, . . . XI are treated as triangles and trapezoids. The final error in this approximation is small for the divisions shown and of course will be further reduced by taking smaller divisions. Any case of varying moment of inertia may be similarly treated.

PROBLEM VIII

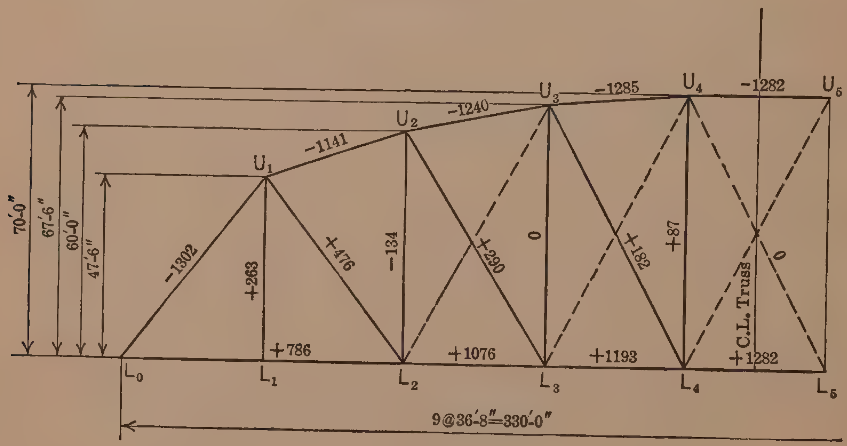


FIG. 50

Problem VIII (Fig. 50) shows camber calculations for the 330 ft. railway truss span of Fig. 111. The calculations for the center deflections (at L_4 and L_5) are made (*a*) for maximum loading $(D+L+I)$ and (*b*) for $[D+\frac{1}{2}(L+I)]$. Camber is usually provided just sufficient to offset (*b*). The last two columns of the table show the necessary modifications in length (1) when camber is provided by changing the lengths of all members and (2) when it is secured by changing the top chord members only. An advantage of the latter method lies in the fact that the changes in length, being confined to a few members, can be secured more accurately (due to their greater amount) within the limits of workable dimensions.

Chamber Calculations

330′ Pratt Truss (Problem VIII)

Table (A)—(Fig. 50)

Member	L Inches	A Sq. Inches	S $(D+L+I)$ 1000# Units	u ½ # @ L_4 and L_5	$\frac{L}{AE}$ $\frac{1}{1,000,000}$	$Su\frac{L}{AE}$	(Δ) ⅛″ per Hor. 10 Ft.	(Δ)u	Change in Length for Camber of 2.38″: $\Delta=\frac{SL}{AE}$ $D+\frac{1}{2}(L+I)$	Change in Length for Camber of 2.38″: Δ Same Camber
L_0U_1	720	129.47	−1250	.632	.184	.145	.750	.474	$+\frac{5}{32}$	$+\frac{15}{32}$
U_1U_2	462	90.97	−1113	.644	.169	.121	.492	.310	$+\frac{4}{32}$	$+\frac{9}{32}$
U_2U_3	446	90.97	−1220	.833	.163	.166	.465	.388	$+\frac{4}{32}$	$+\frac{9}{32}$
U_3U_4	442	97.97	−1285	1.052	.150	.207	.460	.484	$+\frac{4}{32}$	$+\frac{9}{32}$
$U_4U_5\left(\frac{L}{2}\right)$	220	97.97	−1282	1.048	.0748	.100	.229	.240	$+\frac{4}{32}$	$+\frac{9}{32}$
L_0L_2	880	53.26	+755	.386	.550	.160	....	....	$-\frac{9}{32}$	Increase in length of upper chord to give same camber as $D+\frac{1}{2}(L+I)$ (.0875″ per hor. 10′ of upper chord).
L_2L_3	440	72.50	+1050	.612	.202	.130	....	....	$-\frac{5}{32}$	
L_3L_4	440	82.50	+1193	.812	.177	.171	....	....	$-\frac{5}{32}$	
$L_4L_5\left(\frac{L}{2}\right)$	220	90.00	+1282	1.048	.0815	.109	....	....	$-\frac{5}{32}$	
U_1L_1	570	19.00	+263	0	1.000	....	....	....	$-\frac{6}{32}$	
U_2L_2	720	42.20	−134	.292	.568	.022	....	....	$-\frac{2}{32}$	
U_3L_3	810	42.20	0	.333	.640	....	....	....	0	
U_4L_4	840	34.20	+87	.072	.819	.005	....	....	$-\frac{2}{32}$	
U_1L_2	720	35.00	+476	.369	.684	.120	....	....	$-\frac{7}{32}$	
U_2L_3	842	26.25	+290	.390	1.068	.121	....	....	$-\frac{7}{32}$	
U_3L_4	920	22.50	+182	.487	1.361	.121	....	....	$-\frac{5}{32}$	
U_4L_5	948	15.75	0	0	2.000	....	....	....		
						1.693	....	1.896		

Deflection for $(D+L+I)=2\times1.693''=3.386''$

Camber for $D+\frac{1}{2}(L+I)=2.380''$ and $\frac{2.38}{12\times330}=\frac{1}{1660}$ (span)

Camber on basis of ⅛″ per hor. 10′ of upper chord $=2\times1.896''=3.792''$

CHAPTER II

GENERAL THEORY OF STATICALLY INDETERMINATE STRESSES

30. Preliminary.—Every structural problem where the number of unknown forces to be found exceeds that which can be obtained by means of the equations of static equilibrium, is said to be statically indeterminate. The setting of the problem has been discussed rather fully from a general standpoint in the Introduction. It was there stated that the necessary additional relations upon which the solution of the problem depends are obtained from the Law of Consistent Deflections. That is to say, in any structure, not only must the requirements of static equilibrium be satisfied, but the resulting elastic deflections must be consistent with the conditions of the problem.

In Chapter I we have shown how the elastic deflections of structures may be obtained by several methods; in this chapter we shall apply these results to the solution of the statically indeterminate problem in general, by means of the principle of consistent elastic deformations or deflections.

Before proceeding further it is well to note explicitly the assumption that underlies the whole development of the theory (as indeed it does other portions of the theory of structures), i.e., that the total effect of a group of forces on the stresses and deflections of a structure is equal to the sum of the effects of the forces taken separately. This is commonly called the "law of superposition."

SECTION 1.—SINGLY INDETERMINATE STRUCTURES

31. General Theory.—Let us consider a continuous girder ABC, Fig. 51a, resting on three rigid supports. We remove the center support and imagine the simple beam AC acted on by the loads $P_1 \ldots P_n$ and an arbitrary upward load P_B at B, Fig. 51b. Clearly, if $P_B = R_B$, the simple beam in (b) becomes the exact equivalent, statically, of the continuous beam in (a). The determining condition to be fulfilled by $P_B = R_B$ is that it shall make the deflection at B equal to zero. We have

$$\delta_B = \int_A^C \frac{M m_B dx}{EI}, \quad \text{and} \quad M = M' + R_B m_B,$$

if M' is the simple beam moment, due to loads P at any point of AC, and $R_B m_B$ is the simple beam moment of any point in AC due to $P_B = R_B$ applied at B.

Hence

$$\delta_B = 0 = \int_A^C \frac{M' m_B dx}{EI} + R_B \int_A^C \frac{m_B{}^2 dx}{EI},$$

from which we get

$$R_B = -\frac{\int_A^C \frac{M' m_B dx}{EI}}{\int_A^C \frac{m_B{}^2 dx}{EI}} = -\frac{\delta'_B}{\delta_{1B}}, \quad \ldots \ldots \quad (25)$$

if δ'_B = deflection at B in simple beam AC due to specified loads, and δ_{1B} = deflection at B in simple beam AC due to an upward load unity applied at B.

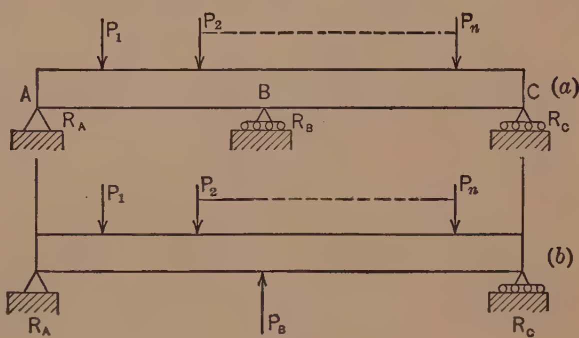


FIG. 51

The physical conception is thus very simple. We imagine the loading applied to the beam with the superfluous reaction removed; this will result in a certain displacement of the reaction point = δ'. We then say that the amount of the true reaction is the magnitude of the force necessary to erase this deflection. A unit load will effect a displacement of δ_1, whence

$$\frac{R}{\text{Unity}} = -\frac{\delta'}{\delta_1}.$$

The same method in principle may be applied to a truss with a redundant member. In the truss of Fig. 52 the tie rod CD may be regarded as a superfluous member. The truss may be rendered statically determinate by the removal of CD, which is accomplished in effect if we cut the member at some point—for convenience very near the end D. When the member is so cut, the cut faces will be displaced relatively by an amount δ' which may be computed by the standard method,

$$\delta' = \sum \frac{S' u L}{AE},$$

where S' = stress in any member of frame, due to given loads, with CD removed (cut), and

u = stress in any member due to a pair of unit forces acting on the cut faces of CD as shown in Fig. 52.

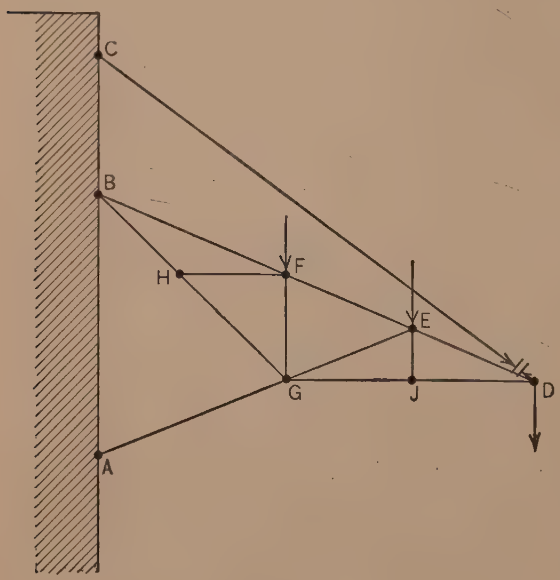


FIG. 52

If now we have a pair of equal and oppositely directed forces acting on the cut faces of CD, *numerically equal to the true stress in CD* when it acts as a part of the frame, this modified structure is evidently statically equivalent to the original, and we must have

$$\delta = \text{relative displacement of cut faces of } CD = 0$$

$$= \sum \frac{SuL}{AE} = \sum \frac{(S' + S_r \cdot u)uL}{AE}$$

$$= \sum \frac{S'uL}{AE} + S_r \sum \frac{u^2L}{AE},$$

and

$$S_r = - \frac{\sum \frac{S'uL}{AE}}{\sum \frac{u^2L}{AE}} = - \frac{\delta'}{\delta_1}, \quad \dots \dots \dots \quad (26)$$

where S_r = magnitude of *true* stress in CD and
S = magnitude of true stress in any member of the frame.

It will be noted that S' for the redundant member is always zero; hence it disappears from the summation in the numerator. We ordinarily say, therefore, that the summation in the denominator includes all members, while that in the numerator includes all except the redundant.

The method is thus seen to be precisely analogous to the case of the redundant reaction for a beam. We cut the superfluous bar, compute the resulting displacement of the faces, and determine the true stress in the bar by the principle that it is equal in magnitude to the force-pair required to bring these faces into contact. A pair of 1# forces will move the faces a distance δ_1 and to move them through the distance δ' will require 1# $\times \frac{\delta'}{\delta_1}$.

The cut in the redundant member may be taken anywhere; if taken sufficiently close to the end, the deformation of the longer portion may be taken as the deformation of the entire member, which simplifies the detail work.

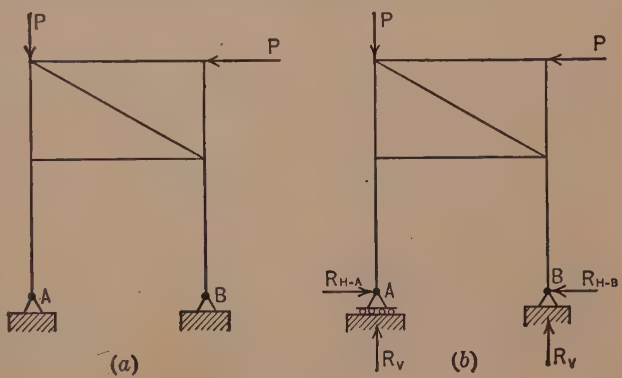


FIG. 53

32. Structures with Members Subjected to Direct Stress and Bending.—The preceding method is easily adapted to the more general case. In the framework of Fig. 53, where some of the members take flexure as well as axial stress, we know that the true stresses must be such as to render the horizontal deflection at A zero, whence we have

$$\delta_{H-A} = \sum \frac{Su_AL}{AE} + \sum \int \frac{Mm_Adx}{EI} = 0$$

$$= \sum \frac{S'u_AL}{AE} + R_{H-A} \sum \frac{u_A{}^2L}{AE} + \sum \int \frac{M'm_Adx}{EI}$$

$$+ R_{H-A} \sum \int \frac{m_A{}^2dx}{EI},$$

whence

$$R_{H-A} = - \frac{\sum \frac{S'u_AL}{AE} + \sum \int \frac{M'm_Adx}{EI}}{\sum \frac{u_A{}^2L}{AE} + \sum \int \frac{m_A{}^2dx}{EI}} = - \frac{\delta'_a + \delta'_b}{\delta_{1a} + \delta_{1b}} = - \frac{\delta'}{\delta_1}, \quad (27)$$

where u_A and m_A = the direct stress and the bending moment, respectively, in any member due to a 1# load acting horizontally inward at A.

δ'_a and δ'_b = the horizontal deflection at A due, respectively, to the axial deformation of all members resulting from the given loads, and the bending of the members due to the given loads.

If we change the subscripts for u and m in the preceding equation from A to r we get the equation for the stress in the redundant member S_r of the structure shown in Fig. 54*b*. Here again u_r and m_r are, respectively, the axial stress in any member and the bending moment at any point of any member due to a pair of unit forces applied in opposite directions to the cut faces of the member.

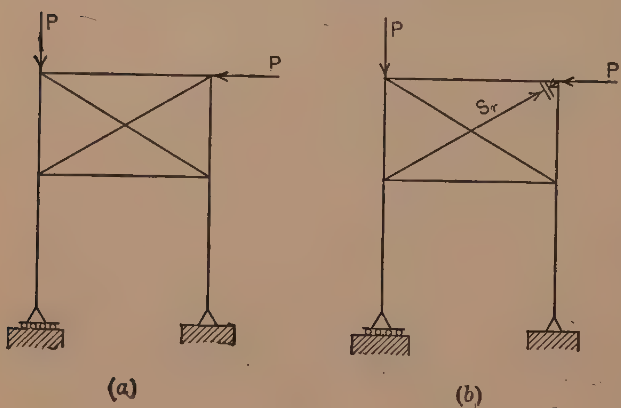


FIG. 54

33. Modification to Include Members Slightly Curved.—Finally, if we have a framework in which some or all of the bars are slightly curved, and in which the section A is not necessarily constant throughout the member, we may write quite generally

$$X_r = \text{redundant quantity, either reaction or stress}$$

$$= -\frac{\sum\int\frac{N'n_r ds}{AE} + \sum\int\frac{M'm_r ds}{EI}}{\sum\int\frac{n_r^2 ds}{AE} + \sum\int\frac{m_r^2 ds}{EI}} = -\frac{\delta_r}{\delta_{1r}}. \quad . \quad . \quad (28)$$

The same remarks regarding the scope of the summation in numerator and denominator apply here as were noted for the simpler case on page 92.

34. General Remarks.—It should be noted that the choice of the reaction or of the member which we treat as redundant is to some extent arbitrary. Usually any reaction or member may be so treated whose

removal leaves a statically determinate stable structure. In the truss of Fig. 52 we might equally well have selected AG, or FE, or any one of several others as the redundant. But we could not so use GE, since its removal leaves an unstable structure and it is therefore *not* a superfluous member. Neither could we select EJ or HF, since for the given loading they are not essential members of the truss; their stress is zero, and their removal still leaves the structure statically undetermined.

The interpretation of the signs requires careful consideration. To restate the general method: We remove the redundant support or member and apply in its place equivalent forces as external loads. If

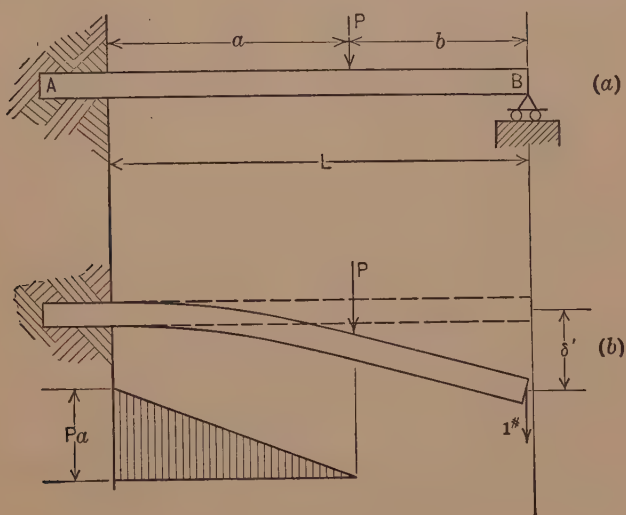


FIG. 55

these are entirely removed, the resulting statically determinate structure will so distort that the points of application of the redundant forces (in case of external reaction there is but a single force) will be displaced an amount δ'. The redundant force X must be such as to cause an equal and opposite deflection $X\delta_1$ i.e.:

$$\delta = 0 = \delta' + X\delta_1.$$

This, of course, assumes that the unit loading producing δ_1 is *opposed* to the displacement δ' and it must always give a positive value of X. If the calculation is carried through as above but with the unit loading applied in the opposite sense, the value so obtained for X will have the same magnitude but opposite sign. A clear understanding of these relations should serve to avoid any confusion as to the sign of X.

35. Examples.—It will aid in fixing the foregoing principles to apply them to a few simple problems.

(a) *Beam Fixed at One End and Freely Supported at the Other.*—(Fig. 55).

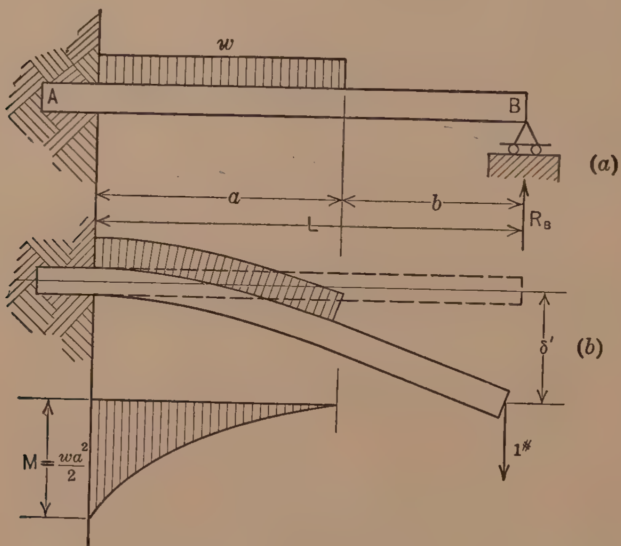


FIG. 56

With origin at B, E and I constant, R_B removed, we have

$$M' = 0, - \text{ from } x = o \text{ to } x = b,$$

$$M' = P(x - b), - \text{ from } x = b \text{ to } x = L,$$

$$m = x, - \text{ (unit load downward).}$$

Then

$$EI\,\delta' = \int_0^L M'm\,dx = \int_b^L P(x - b)x\,dx = \frac{Pa^2}{6}(3L - a)$$

and

$$EI\,\delta_1 = \int_0^L m_2\,dx = \int_0^L x^2\,dx = \frac{L^3}{3},$$

whence

$$R_B = -\frac{\delta'}{\delta_1} = -\frac{\frac{Pa^2}{6}(3L - a)}{\frac{L^3}{3}} = -\frac{P}{2}\frac{a^2}{L^3}(3L - a).$$

The minus sign means that R_B acts oppositely to the unit load.

Consider the same beam with uniform load extending a distance a from the fixed end (Fig. 56). Proceeding as before

$$M' = \begin{cases} o, & -\ 0 \text{ to } b \\ \dfrac{w(x-b)^2}{2}, & -\ b \text{ to } L \end{cases}$$

$$EI\,\delta' = \int_0^L M'm\,dx = \int_b^L \frac{w(x-b)^2}{2}x\,dx = \frac{wa^3}{24}(4L-a)$$

$$EI\,\delta_1 = \frac{L^3}{3},$$

and

$$R_B = -\frac{\delta'}{\delta_1} = -\frac{w}{8}\left(\frac{a}{L}\right)^3(4L-a).$$

When $a = L$

$$R_B = -\tfrac{3}{8}wL.$$

It is not necessary to use R_B as the redundant; we may take the end moment, M_A, equally well. In this case the statically determinate structure is as shown in Fig. 57. The fundamental equation is

$$M_A = -\frac{\alpha'_A}{\alpha_{1A}} = -\frac{\int M'm\,dx}{\int m^2dx},$$

where m is the moment at any section due to a unit *couple* applied at A. We then have

$$M' = \begin{cases} \dfrac{wa^2x}{2L}, & \ldots\ 0 \text{ to } b \\ \dfrac{wa^2x}{2L} - \dfrac{w(x-b)^2}{2}, & \ldots\ b \text{ to } L, \end{cases}$$

$$m = -\frac{x}{L},$$

and

$$EI\alpha'_A = \int^L Mm\,dx = -\int_0^L \frac{wa^2}{2L^2}x^2dx + \int_b^L \frac{w(x-b)^2x\,dx}{2}$$

$$= -\frac{wa^2}{24L}(2L-a)^2,$$

$$\alpha_{1A} = \frac{L}{3},$$

whence

$$M_A = \frac{wa^2}{8L^2}(2L - a)^2.$$

The plus sign indicates that M_A acts in the same direction as the dummy unit couple.

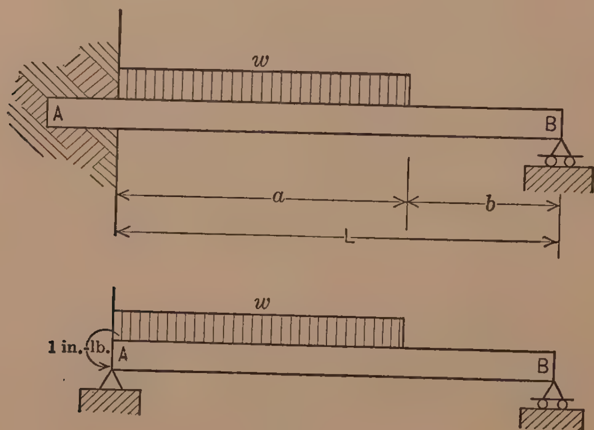
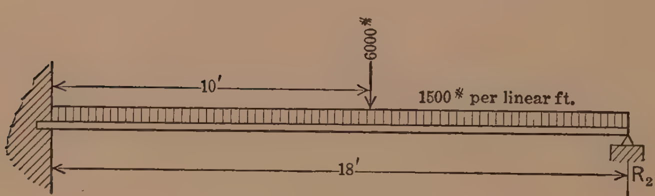


FIG. 57



20″ − 65# *I* beam.
Required: Value of R_2
FIG. 57*a*

$$R_2 = \frac{-\delta'}{\delta_1} = \frac{-\int \frac{Mmdx}{EI}}{\int \frac{m^2dx}{EI}} = \frac{-\int Mmdx}{\int m^2dx}$$

$$R_2 = \frac{\int_0^8 \frac{1500x^3dx}{2} + \int_8^{18}\left[\frac{1500x^2}{2} + 6000(x-8)\right]xdx}{\int_0^{18} x^2dx} = -\frac{23,969,000}{1,944}$$

$R_2 = -12,310$#

The unit load, was assumed to act downward, hence the negative sign means that R_2 acts upward.

We may check this from the preceding result:

$$-M_A = R \cdot L - \frac{wa^2}{2} = \frac{w}{8}\frac{a^3}{L^3}(4L - a) - \frac{wa^2}{2} = -\frac{wa^2}{8L^2}(2L - a)^2.$$

Fig. 57a shows a numerical example.

(b) *Continuous Girder of Two Equal Spans, Uniform Load* (Fig. 58).

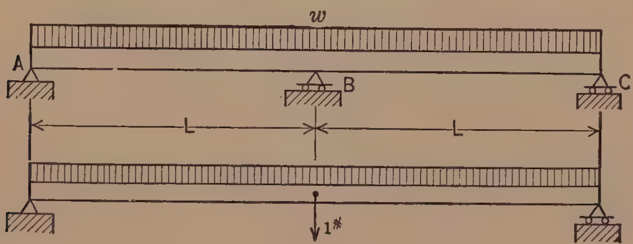


FIG. 58

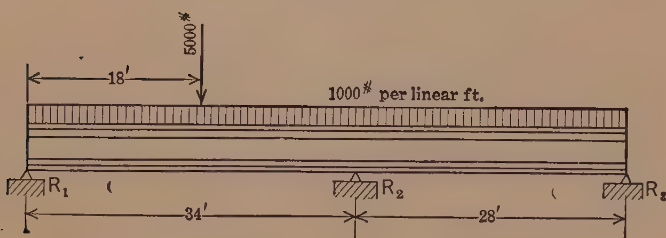


FIG. 58a

SECTION

1 Web plate 30″ × ½″ 2 Angles 6″ × 4″ × ⅝″

2 Cover plates 14″ × ⅝″

Required: Value of R_2

$$R_2 = -\left[\frac{\left\{\int_0^{18}\left(3455x - \frac{x^2}{2}\right).451x dx + \int_{18}^{34}\left[34.55x - \frac{x^2}{2} - 5(x - 18)\right]\right.\left. .451x dx + \int_0^{28}\left(32.45x - \frac{x^2}{2}\right).549x dx\right\}}{\int_0^{34}.451^2x^2dx + \int_0^{28}.549^2x^2dx}\right]$$

$R_2 = -\dfrac{208{,}000}{4865} \times 1000 = -427{,}500\#$; the negative sign means that R_2 acts upward.

We treat the center support as redundant and take origin at A.

$$M' = wLx - \frac{wx^2}{2}\Big|_0^L, \quad m = \frac{x}{2}\Big|_0^L,$$

and

$$\tfrac{1}{2}EI\delta'_B = \int_0^L M'm\,dx = \int_0^L \left(\frac{wLx^2}{2} - \frac{wx^3}{4}\right)dx = \frac{5}{48}wL^4,$$

$$\tfrac{1}{2}EI\delta_{1B} = \int_0^L m^2dx = \int_0^L \frac{x^2}{2}dx = \frac{L^3}{6},$$

whence

$$R_B = -\frac{\delta'_B}{\delta_{1B}} = -\frac{\frac{5}{48}wL^4}{\frac{L^3}{6}} = -\frac{5}{8}wL,$$

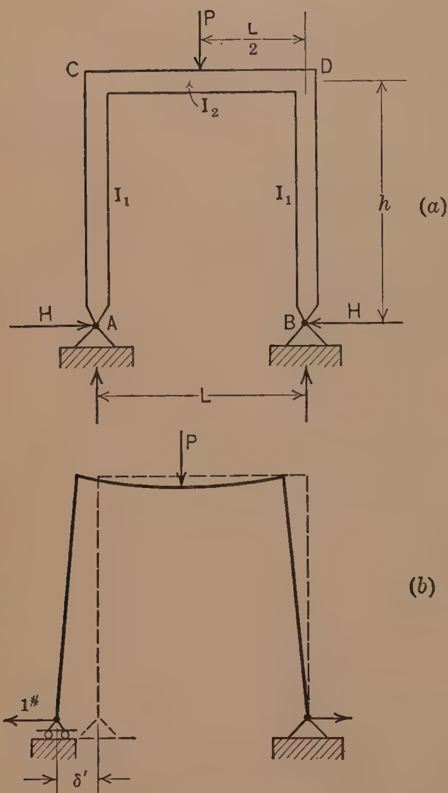


FIG. 59

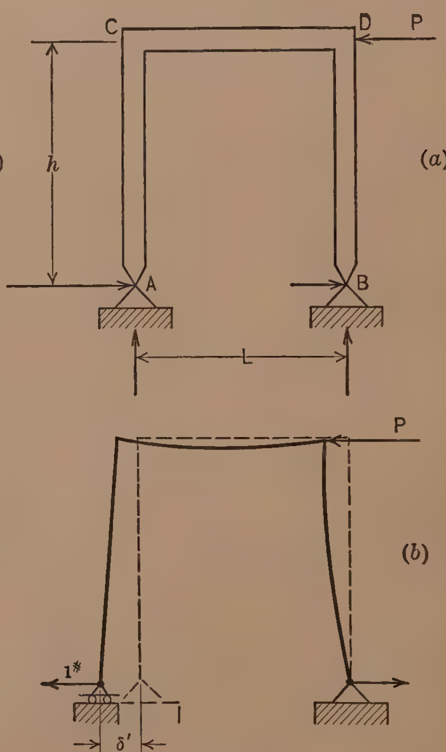


FIG. 60

which is the well-known formula for the center reaction in a continuous beam of two equal spans, loaded uniformly.

The same general method may be applied to a two-span continuous girder of unequal spans and any loading. Fig. 58a shows a numerical case.

(c) *Portal Frame* (Fig. 59).

We treat the horizontal reaction as redundant, and neglect the effect of shortening in CD due to H (generally exceedingly small in such a frame). The fundamental equation is,

$$H = -\frac{\delta'}{\delta_1} = -\frac{\sum \int \frac{M'm\,dx}{EI}}{\sum \int \frac{m^2 dx}{EI}},$$

$$\sum \int \frac{M'm\,dx}{EI} = 2 \times \frac{1}{EI_2}\int_0^{\frac{L}{2}} \frac{Px}{2}\cdot h\,dx = \frac{PhL^2}{8EI_2},$$

$$\sum \int \frac{m^2 dx}{EI} = 2\int_0^h \frac{x^2 dx}{EI_1} + \int_0^L \frac{h^2 dx}{EI_2} = \frac{2}{3}\frac{h^3}{EI_1} + \frac{h^2 L}{EI_2},$$

whence

$$H = -\frac{\frac{PhL^2}{8EI_2}}{\frac{2}{3}\frac{h^3}{EI_1} + \frac{h^2 L}{EI_2}}.$$

The unit loading was applied outwardly; the minus sign shows that H acts inwardly.

We may take the same frame with horizontal load at top (Fig. 60). Again neglecting axial shortening, we have

$$H_A = -\frac{\delta'_A}{\delta_{1A}} = -\frac{\sum \int \frac{M'm\,dx}{EI}}{\sum \int \frac{m^2 dx}{EI}}$$

$$M' = 0 \text{ for } AC; \quad = \frac{Ph}{L}\Big|_0^L \text{ for } CD, \text{ origin at } C,$$

and

$$= Px\Big|_0^h \text{ for } DB, \text{ origin at } B,$$

m is same as for preceding case.

We then have

$$\sum \int \frac{M'm\,dx}{EI} = \int_0^L \frac{Phx}{EI_2 L}\cdot h\,dx + \int_0^h \frac{Px^2 dx}{EI_1} = \frac{Ph^2 L}{2EI_2} + \frac{Ph^3}{3EI_1},$$

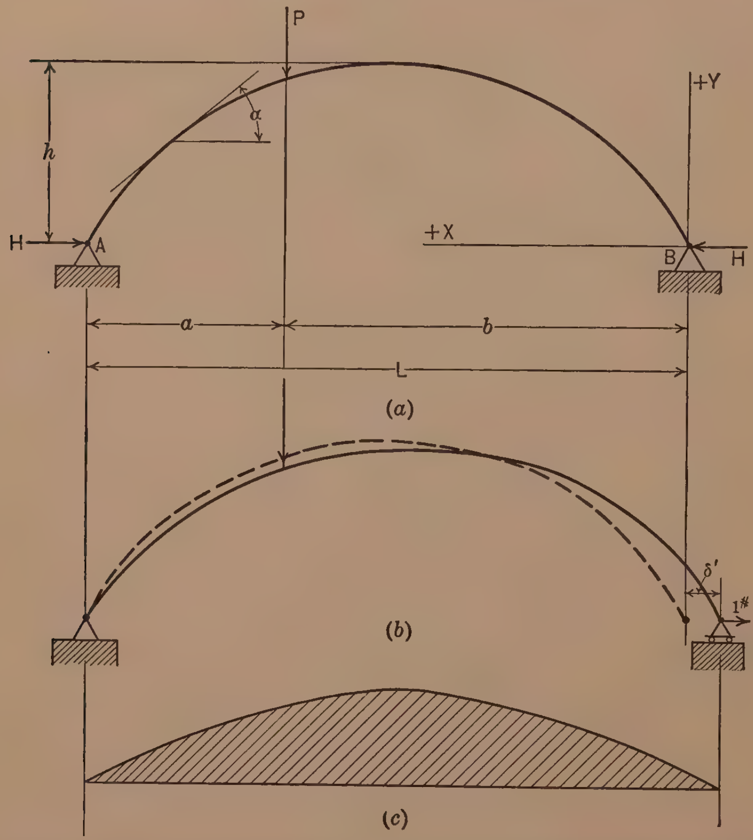
whence

$$H_A = -\frac{\frac{Ph^2 L}{2EI_2} + \frac{Ph^3}{3EI_1}}{\frac{h^2 L}{EI_2} + \frac{2}{3}\frac{h^3}{EI_1}} = -\frac{P}{2},$$

the usual approximate formula.

If the member CD should develop an appreciable axial deformation, or if the load P should be applied to the column BD at an intermediate point, the above result would no longer hold.

(*d*) *Two-hinged Arch Rib with Parabolic Axis* (Fig. 61).



Deflection line for arch axis under pair of unit horizontal loads at supports acting inward = (to some scale) influence line for H.

FIG. 61

Taking positive direction of coordinates upwards and to the left, the equation of the parabolic axis when referred to end B is

$$y = 4h\left(\frac{x}{L} - \frac{x^2}{L^2}\right).$$

The cross-section of an arch rib usually increases toward the support; a common assumption which gives a very satisfactory approximation

in most cases is that I varies as secant α (α = angle of inclination of arch axis with axis of x). In such case, if $I_c = I$ at crown,

$$I = I_c \sec. \alpha, \text{ and } \frac{ds}{I} = \frac{ds \cos \alpha}{I_c} = \frac{dx}{I_c},$$

and the deflection equations are considerably simplified. Further, the terms representing axial thrust are, in all ordinary cases, quite small and may be neglected without serious error.

Making these simplifications and taking the horizontal thrust as the redundant, we have,

$$H = -\frac{\delta'_B}{\delta_{1B}} = -\frac{\int \frac{M'mds}{EI}}{\int \frac{m^2ds}{EI}} = -\frac{\int M'mdx}{\int m^2dx},$$

$$M' = \begin{cases} \left.\frac{Pa}{L}x\right|_0^b \\ \left.\frac{Pax}{L} - P(x-b)\right|_b^L \end{cases}; \quad m = y = 4h\left(\frac{x}{L} - \frac{x^2}{L^2}\right),$$

$$\int_0^L M'mdx = \int_0^L \frac{4Pahx}{L}\left(\frac{x}{L} - \frac{x^2}{L^2}\right)dx - \int_b^L \frac{4Ph}{L^2}(Lx - x^2)(x - b)dx,$$

and

$$\int_0^L m^2dx = \int_0^L 16h^2\left(\frac{x}{L} - \frac{x^2}{L^2}\right)^2 dx.$$

These integrals are easily evaluated:

$$\int_0^L \frac{4Pahx}{L}\left(\frac{x}{L} - \frac{x^2}{L^2}\right)dx = \frac{PahL}{3},$$

$$\int_b^L \frac{4Ph}{L^2}(Lx - x^2)(x - b)dx = \frac{4Ph}{L^2}\int_0^L (x^2L - x^3 - Lbx + bx^2)dx$$

$$= \frac{Ph}{3}\left(\frac{a}{L}\right)^2[a(2L - a)],$$

$$\int_0^L 16h^2\left(\frac{x}{L} - \frac{x^2}{L^2}\right)^2 dx = \frac{8}{15}h^2L.$$

$$\therefore \frac{\int M'mdx}{\int m^2dx} = \frac{\frac{PahL}{3} - \frac{Ph}{3}\left(\frac{a}{L}\right)^2[a(2L - a)]}{\frac{8}{15}h^2L},$$

whence

$$H = -\frac{5}{8}\frac{PL}{h}\left[\frac{a}{L} - \frac{a^3}{L^3}\left(2 - \frac{a}{L}\right)\right].$$

If $\frac{a}{L} = k$, we have

$$H = -\frac{5}{8}\frac{PL}{h}(k - 2k^3 + k^4).$$

Fig. 61*b* indicates the distortion of the arch when H is removed.

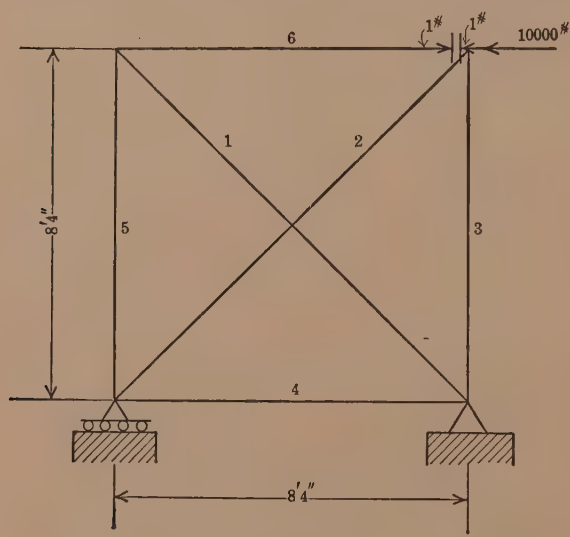


Fig. 62

TABLE A

Member	A	L	S'	$\frac{S'L}{A}$	u	$\frac{SL\cdot u}{A}$	$\frac{u^2L}{A}$	S_ru	$S=S'+S_ru$
1	3	141	0	....	−1.41	....	94.0	+7000	+7000
2	3	141	−14,100	−667,000	−1.41	+940,000	94.0	+7000	−7000
3	10	100	+10,000	+100,000	+1	+100,000	10.0	−5000	+5000
4	3	100	+10,000	+333,000	+1	+333,000	33.3	−5000	+5000
5	10	100	0	....	+1	....	10.0	−5000	−5000
6	3	100	0	....	−1	....	33.3	+5000	+5000
						+1,373,000	+274.6		

$$S_r = -\frac{\delta'}{\delta_1} = \frac{-1{,}373{,}000}{274.6} = -5000^{\#} = \text{Stress in } S_6.$$

Since the unit loading is such as to produce tension in S_6, the negative sign means that the true stress is compression.

(*e*) *Truss with Redundant Member.*

The general method of procedure for this problem has been previously indicated. Fig. 62 and Table A show the full detail of a very simple numerical example.

(*f*) *Continuous Truss.*

This is precisely similar to the solid girder if we use the truss deflection formula instead of the beam deflection formula.

We have

$$R_e = -\frac{\delta'_e}{\delta_{1e}} = -\frac{\sum \frac{S'u_eL}{AE}}{\sum \frac{u_e^2L}{AE}}.$$

The notation is self-explanatory and the detail involves nothing but a straightforward application of the deflection formulas.

(*g*) *The Spandrel-braced Arch.*

The horizontal thrust is the redundant; if R_{rH} be removed the

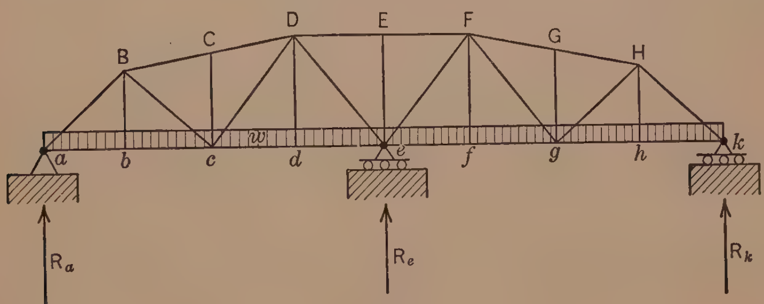


FIG. 63

point g will deflect to the right a distance δ' under the action of the load w. If u_g is the stress in any member due to a unit horizontal load acting outwardly at g,

$$\delta'_g = \sum \frac{S'u_gL}{AE}.$$

The true horizontal thrust is the force required to produce an equal and oppositely directed deflection. A 1# load will deflect g

$$\delta_{1g} = \sum \frac{u_g^2L}{AE},$$

and

$$R_H = -\frac{\delta'_g}{\delta_{1g}} = -\frac{\sum \frac{S'u_gL}{A}}{\sum \frac{u_g^2L}{A}}.$$

Fig. 64*a* and table show a numerical case.

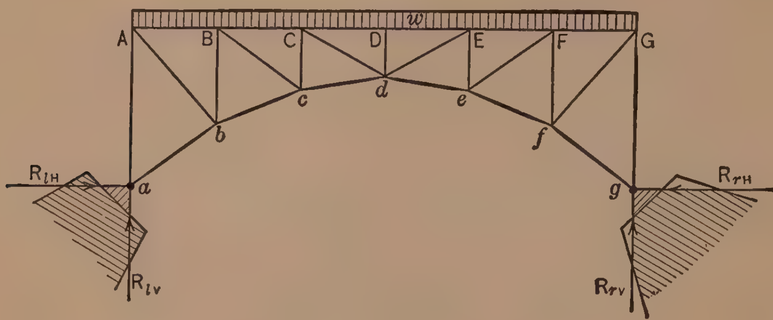


FIG. 64

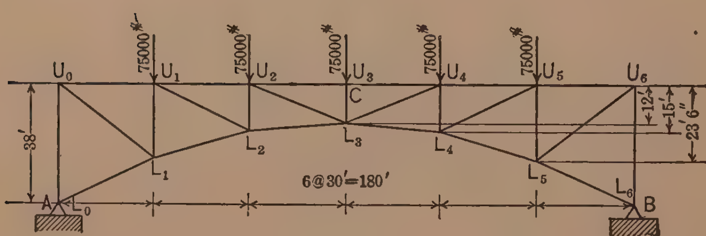


FIG. 64(a)

TABLE A

Member	Length (L) Inches	Stress (S) Pounds	Area (A) Sq. In.	$\frac{SL}{EA}$	u	$\frac{SuL}{EA}$	$\frac{u^2L}{EA}$
U_0U_1	360	−239,000	10.00	− .307	− .617	+ .1892	.00000045
U_1U_2	360	−600,000	10.00	− .771	−1.533	+1.182	.00000282
U_2U_3	360	−844,000	10.00	−1.085	−2.165	+2.350	.00000561
L_0L_1	400	0	20.00	000	+1.11	000	.00000082
L_1L_2	374	+249,000	18.00	+ .1845	+1.678	+ .3095	.00000195
L_2L_3	362	+603,000	16.00	+ .4325	+2.55	+1.102	.00000490
U_0L_1	457	+304,000	10.00	+ .496	+ .785	+ .389	.00000090
U_1L_2	402	+404,000	8.00	+ .725	+1.025	+ .744	.00000176
U_2L_3	388	+274,500	6.00	+ .634	+ .680	+ .431	.00000099
U_0L_0	456	−187,500	10.00	− .3055	− .483	+ .1473	.00000035
U_1L_1	282	−255,000	8.00	− .3215	− .458	+ .1473	.00000025
U_2L_2	180	−170,500	6.00	.1827	− .253	+ .0462	.00000006
U_3L_3	144	−37,500*	2.00*	.0964	.000	.0000	
						7.0375	.00002086

*One-half actual value.

$$H = -\frac{7.038}{.000021} = -350{,}000^{\#}$$

The minus sign indicates that H acts inward (opposite to deflection δ').

(*h*) *The Framed Bent* (*Truss and Beam Combination*) (Fig. 65).

The columns ACD and BKH are continuous over points C and K. All other members take axial stress only. We take R_{lH} as the redundant (we may take R_{rH} equally well) and from the discussion which has preceded, we may write at once,

$$\delta'_A = \sum \frac{S'uL}{AE} + \sum \int_0^L \frac{Mmdx}{EI},$$

and

$$\delta_{1A} = \sum \frac{u^2L}{AE} + \sum \int_0^L \frac{m^2dx}{EI},$$

whence

$$R_{lH} = -\frac{\delta'_A}{\delta_{1A}} = -\frac{\sum \frac{S'uL}{A} + \sum \int_0^L \frac{M'mdx}{I}}{\sum \frac{u^2L}{A} + \sum \int_0^L \frac{m^2dx}{I}}.$$

Any truss and beam combination is analyzed similarly. Fig. 65*a* is a simple example of a "King post" truss. Member ACB is continuous over joint C. If the member (1) is taken as the redundant, and we imagine it cut at the upper end, we have

$$S_{①} = -\frac{\delta'}{\delta_1} = -\frac{\sum \frac{S'uL}{AE} + \sum \frac{M'mdx}{EI}}{\sum \frac{u^2L}{AE} + \sum \frac{m^2dx}{EI}}. \quad . \quad . \quad . \quad (a)$$

In the substitute statically determinate structure, it is obvious that the load P is carried to the supports entirely by bending in AB; therefore the term $\frac{S'uL}{AE}$ vanishes. It is further clear from the figure that $M' = -Pm$. We shall then have

$$S_{①} = P\frac{\int \frac{m^2dx}{I}}{\sum \frac{u^2L}{A} + \int \frac{m^2dx}{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$\int_0^L \frac{m^2dx}{I} = 2\int_0^{\frac{L}{2}} \frac{x^2}{2}\frac{dx}{I} = \frac{L^3}{48I} = 1340.$$

$$\sum \frac{u^2L}{AE} \text{ (from Table A)} = 807.$$

$$\therefore\ S_{①} = \frac{1340}{1340 + 807} \times 10000 = 6200\#. \quad . \quad . \quad . \quad (c)$$

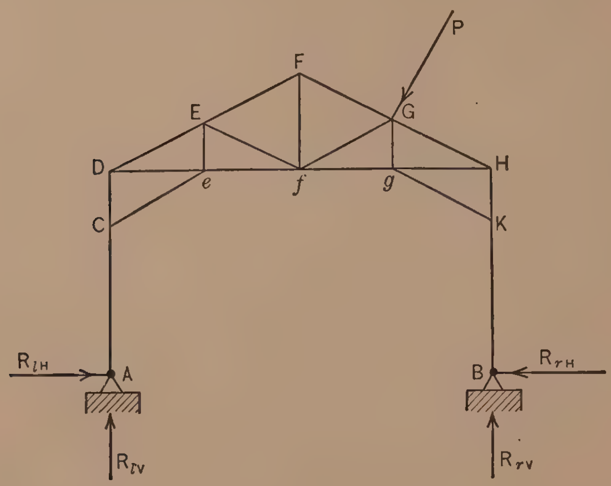


FIG. 65

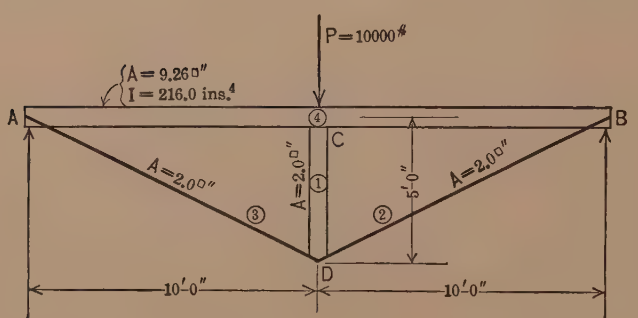


FIG. 65(*a*)

TABLE A

Member	A	L	$\frac{L}{A}$	u	u^2	$\frac{u^2L}{A}$
AB	9.26	240	26.0	−2.0	4.0	104.0
CD	2.0	60	30.0	−1.0	1.0	30.0
$AD = BD$	2.0	134.5	67.3	+2.24	5.0	336.5

$$\sum \frac{u^2L}{A} = 807.0$$

The positive sign indicates that $S_{①}$ is of the same sign as $u_{①}$; i.e., compression.

It may be interesting to view the problem from a different standpoint. We have noted that when member CD is cut, the load is carried entirely by beam action in ACB. But if the framework were rendered determinate by breaking up the member ACB into separate members AC and CB, we should have the entire load carried by the simple truss $ACBD$. In the actual framework there is a combined action, and the problem is solved if we can answer the question: How much load is carried by truss action in $ACBD$, and how much by beam action in ACB? Now the deflection of the true truss (no continuity in ACB at C) for a unit load at C is $\sum \frac{u^2L}{AE}$; the first term in the denominator of (b). Likewise the deflection of the simple beam AB for a unit load at C is $\int \frac{m^2dx}{EI}$, the numerator of the fraction in the right-hand member of (b) and also the second term in the denominator. If we call

$$\frac{1}{\int \frac{m^2dx}{EI}} = r_b$$

= the coefficient of rigidity of the beam AB with respect to a vertical load at C, and define the rigidity coefficient r_t for the truss similarly, we may write Eq. (b)

$$S_{①} = P\frac{\frac{1}{r_b}}{\frac{1}{r_t} + \frac{1}{r_b}} = \frac{r_t}{r_t + r_b}P \quad . \; . \; . \; . \; . \; . \; . \quad (d)$$

Now, the stress $S_{①}$ measures the amount of the load which the truss carries and $P - S$ the portion of the load carried by AB acting as a beam. We may easily show from (d) that $\frac{S_{①}}{P - S_{①}} = \frac{r_t}{r_b}$; i.e., the relative distribution of the load through the beam and through the truss is in proportion to their relative rigidities.

This result illustrates a very fundamental principle in the theory of redundant structures, usually termed the "principle of rigidities." In general, in transmitting a load to its final support, the stress tends to follow the most rigid path.

36. Summary.—We may summarize briefly:

Any structure containing a single redundant may be reduced to an equivalent structure from which the redundant has been removed and

in its place a statically equivalent loading applied. If the redundant is a simple support this loading is a single force; if the redundant is a superfluous bar, the loading is a pair of equal and opposite forces; if the redundant is a "fixed-end" reaction, the equivalent loading is a couple. The problem is to find this unknown loading which is numerically equal to the redundant quantity, and which we may designate in general by X. The statically determined structure which results from the removal of the redundant we shall for brevity call the base-system.

If now we imagine X to be entirely removed and the specified loading to be applied to the base-system, there will result a certain displacement of the point of application of X which we call δ' and which is easily calculated from the fundamental formulas. (When the redundant is a pair of forces, the displacement of the point of application of X is to be interpreted as the relative movement of one of the equal and opposite forces with respect to the other.)*

If we next imagine the specified loading removed and X alone to be applied to the base-system, we shall find that the application point of X is displaced an amount $X\delta_1$, if δ_1 is the displacement for the case X = unity, and no other loads act on the base-system. But, if X is to be equivalent to the actual redundant reaction or redundant stress, then (in all such cases as we have been considering) these two deflections must be equal and opposite, i.e.,

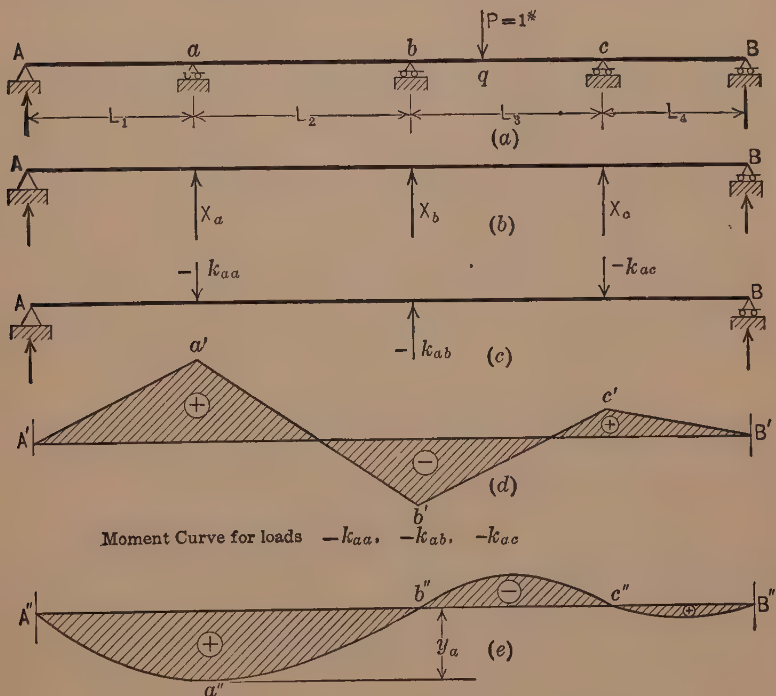
$$\delta' + X\delta_1 = 0; \quad X = -\frac{\delta'}{\delta_1}.$$

It is important for the student to recognize clearly that this simple equation applies directly to a great variety of problems—simple beams, simple trusses, and what we may call truss and beam combinations (as case "h"). It should also be noted that δ is here used as a *general* term for displacement, either *linear* or *angular*.

* If the redundant is a moment, the displacement is a rotation, and in such case the use of the term "*point* of application" is open to criticism. If for any moment, however, we take a statically equivalent couple consisting of a pair of indefinitely large forces with a correspondingly small arm, we may approach as nearly as we please to the condition of a moment applied at a point, and with this interpretation we may properly speak of the rotation of the "point." (See Professor Geo. F. Swain, "A New Principle in the Theory of Structures," Trans. A. S. C. E. Vol. LXXXIII, p. 622, et seq.) At any rate, the gain in simplicity of statement would seem to make this terminology defensible in this case.

SECTION II.—DEVELOPMENT OF FORMULAS FOR STRUCTURES OF ANY DEGREE OF STATICAL INDETERMINATENESS

37. General Equations.—The preceding method is readily extended to structures with any number of redundants. To fix ideas we shall first consider a continuous girder with four spans (Fig. 66). If we



Moment curve for a load diagram $= A'a'b'c'B'$
$= EI \times$ Deflection curve for loading k_{aa}, k_{ab}, k_{ac}
$=$ Influence line for X_a (if y_a be made $= 1$).

FIG. 66

remove R_a, R_b and R_c we shall get, from the given loading, deflections δ'_a, δ'_b and δ'_c at a, b, and c. Now R_a, R_b and R_c must be so adjusted that the resultant deflection at each of these points is zero. If in general we let δ_{mn} = deflection at m due to a unit loading at n in the base-system, we may express the above conditions mathematically in the equations

$$R_a\delta_{aa} + R_b\delta_{ab} + R_c\delta_{ac} + \delta'_a = 0,$$

and two similar equations.

In the general case of n redundants we have

$$\left.\begin{array}{l} X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} + \ldots X_n\delta_{an} + \delta'_a = 0 \\ X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} + \ldots X_n\delta_{bn} + \delta'_b = 0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ X_a\delta_{na} + X_b\delta_{nb} + X_c\delta_{na} + \ldots X_n\delta_{nn} + \delta'_n = 0 \end{array}\right\} \quad \ldots\ldots\ldots \quad (29)$$

This always gives n equations by which we may determine the n statically indeterminate quantities. The coefficients of X in the above equations, as well as the constant terms, are deflections of the statically determined base-system and are all readily obtained. Considerable simplification is possible if we remember that $\delta_{mn} = \delta_{nm}$ from Maxwell's theorem of reciprocal deflections. In general

$$\delta'_r = \sum \frac{S'u_rL}{AE} + \sum \int \frac{M'm_r dx}{EI},$$

and

$$\delta_{rq} = \sum \frac{u_q u_r L}{AE} + \sum \int \frac{m_q m_r dx}{EI}.$$

For a structure with a large number of redundants, general formulas for X in which the value may be obtained by direct substitution are quite out of the question; indeed when the number is more than two or three, the unwieldiness of the algebraic forms renders their practical usefulness doubtful. In such cases it is generally simpler to substitute the numerical values of the δ's in equations (29) and solve the resulting numerical equations for the X's.

The two examples following will make clear the application of the general method.

37a. Examples.

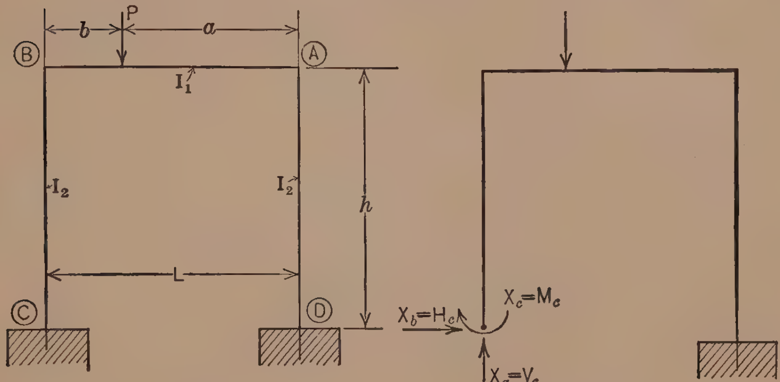
Problem 1 is a quadrangular frame with columns fixed at base and is therefore triply indeterminate. Fig. 67a shows the frame and loading and Fig. 67b shows the base structure with external loadings applied equivalent to the redundant reactions. The solution follows.

The fundamental equations for the statically undetermined quantities are:

$$\delta_a = 0 = \delta'_a + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac}. \quad \ldots\ldots\ldots \quad (a)$$

$$\delta_b = 0 = \delta'_b + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc}. \quad \ldots\ldots\ldots \quad (b)$$

$$\delta_c = 0 = \delta'_c + X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc}. \quad \ldots\ldots\ldots \quad (c)$$



(a)
FIG. 67(a)

(b)
FIG. 67(b)

To evaluate the δ's we proceed as follows (axial distortion is neglected):—

$$\delta'_a=\sum\int\frac{M'm_a dx}{EI}=-\int_b^L\frac{P(x-b)x\,dx}{EI_1}-\int_0^h\frac{PaL\,dx}{EI_2}=\frac{-P}{6EI_1}[a^2(3L-a)]-\frac{PahL}{EI_2}$$

$$\delta'_b=\sum\int\frac{M'm_b dx}{EI}=\int_b^L\frac{P(x-b)h\,dx}{EI_1}+\int_0^h\frac{Pax\,dx}{EI_2}=\frac{Pah}{2E}\left[\frac{a}{I_1}+\frac{h}{I_2}\right]$$

$$\delta'_c=\sum\int\frac{M'm_c dx}{EI}=-\int_b^L\frac{P(x-b)dx}{EI_1}-\int_0^h\frac{Pa\,dx}{EI_2}=-\frac{Pa}{2E}\left[\frac{a}{I_1}+\frac{2h}{I_2}\right]$$

$$\delta_{aa}=\sum\int\frac{m_a^2dx}{EI}=\int_0^L\frac{x^2dx}{EI_1}+\int_0^h\frac{L^2dx}{EI_2}=\frac{L^2}{3E}\left[\frac{L}{I_1}+\frac{3h}{I_2}\right]$$

$$\delta_{bb}=\sum\int\frac{m_b^2dx}{EI}=2\int_0^h\frac{x^2dx}{EI_2}+\int_0^L\frac{h^2dx}{EI_1}=\frac{h^2}{3E}\left[\frac{2h}{I_1}+\frac{3L}{I_1}\right]$$

$$\delta_{cc}=\sum\int\frac{m_c^2dx}{EI}=2\int_0^h\frac{dx}{EI_2}+\int_0^L\frac{dx}{EI_1}=\left[\frac{2h}{EI_2}+\frac{L}{EI_1}\right]$$

$$\delta_{ab}=\sum\int\frac{m_b m_a dx}{EI}=-\int_0^L\frac{hx\,dx}{EI_1}-\int_0^h\frac{Lx\,dx}{EI_2}=\frac{-Lh}{2E}\left[\frac{L}{I_1}+\frac{h}{I_2}\right]=\delta_{ba}$$

$$\delta_{ac}=\sum\int\frac{m_c m_a dx}{EI}=\int_0^L\frac{x\,dx}{EI_1}+\int_0^h\frac{L\,dx}{EI_2}=\frac{L}{2E}\left[\frac{L}{I_1}+\frac{2h}{I_2}\right]=\delta_{ca}$$

$$\delta_{bc}=\sum\int\frac{m_c m_b dx}{EI}=-2\int_0^h\frac{x\,dx}{EI_2}-\int_0^L\frac{h\,dx}{EI_1}=\frac{-h}{E}\left[\frac{L}{I_1}+\frac{h}{I_2}\right]=\delta_{cb}$$

With all constants and all coeficients of the quantities X thus determined, the equations (a), (b) and (c) are readily solved. In any case ordinarily arising in practice h and L are known in advance and I_1 and I_2 are either known or relative values are assumed, and the algebraic detail then becomes quite simple.

The solution to obtain general expressions for the unknowns is quite lengthy and tedious and is seldom of enough advantage to justify itself in any individual problem. As an illustration of the general method we will indicate the process for obtaining the general formula for the horizontal reaction X_b.

In such case where no more than three equations are involved, it is best to first write out the general expression for X_b from equations (a), (b) and (c). This may be done by ordinary elimination, but it is most easily effected by use of determinants. We have:

$$X_b = \frac{\begin{vmatrix} \delta_{aa} & -\delta'_a & \delta_{ac} \\ \delta_{ba} & -\delta'_b & \delta_{bc} \\ \delta_{ca} & -\delta'_c & \delta_{cc} \end{vmatrix}}{\begin{vmatrix} \delta_{aa} & \delta_{ab} & \delta_{ac} \\ \delta_{ba} & \delta_{bb} & \delta_{bc} \\ \delta_{ca} & \delta_{cb} & \delta_{cc} \end{vmatrix}}, \quad \dots\dots \quad (d)$$

$$= -\frac{\delta'_a(\delta_{ba}\delta_{cc} - \delta_{bc}\delta_{ac}) + \delta'_b(\delta_{aa}\delta_{cc} - \delta^2 ac) + \delta'_c(\delta_{aa}\delta_{bc} - \delta_{ac}\delta_{ab})}{\delta_{ab}(\delta_{ba}\delta_{cc} - \delta_{bc}\delta_{ac}) + \delta_{bb}(\delta_{aa}\delta_{cc} - \delta^2 ac) + \delta_{cb}(\delta_{aa}\delta_{bc} - \delta_{ac}\delta_{ab})}, \quad (e)$$

if (d) is expanded by means of the minors of the terms of the middle column.

The coefficients of the δ''s are (dropping the constant E),

$$\delta_{ba}\delta_{cc} - \delta_{bc}\delta_{ac} = \frac{Lh}{2}\left[\frac{L}{I_1} + \frac{h}{I_2}\right]\left\{-\left[\frac{L}{I_1} + \frac{2h}{I_2}\right] + \left[\frac{L}{I_1} + \frac{2h}{I_2}\right]\right\} = 0$$

$$\delta_{aa}\delta_{cc} - \delta^2 ac = L^2\left[\frac{L}{I_1} + \frac{2h}{I_2}\right]\left\{\frac{1}{3}\left[\frac{L}{I_1} + \frac{3h}{I_2}\right] - \frac{1}{4}\left[\frac{L}{I_1} + \frac{2h}{I_2}\right]\right\}$$

$$= \frac{L^2}{12}\left[\frac{L}{I_1} + \frac{2h}{I_2}\right]\left[\frac{L}{I_1} + \frac{6h}{I_2}\right]$$

$$\delta_{aa}\delta_{bc} - \delta_{ac}\delta_{ab} = L^2h\left[\frac{L}{I_1} + \frac{h}{I_2}\right]\left\{-\frac{1}{3}\left[\frac{L}{I_1} + \frac{3h}{I_2}\right] + \frac{1}{4}\left[\frac{L}{I_1} + \frac{h}{I_2}\right]\right.$$

$$= -\frac{L^2h}{12}\left[\frac{L}{I_1} + \frac{h}{I_2}\right]\left[\frac{L}{I_1} + \frac{6h}{I_2}\right]$$

Substituting in (e) these coefficients and the values of the δ''s themselves, we have $\left(\text{after cancelling the common factor } \left[\frac{L}{I_1} + \frac{6h}{I_2}\right]\right)$:—

$$X_b = -\frac{Pa}{2h} \cdot \frac{\left[\frac{a}{I_1} + \frac{h}{I_2}\right]\left[\frac{L}{I_1} + \frac{2h}{I_2}\right] - \left[\frac{a}{I_1} + \frac{2h}{I_2}\right]\left[\frac{L}{I_1} + \frac{h}{I_2}\right]}{\left[\frac{L}{I_1} + \frac{2}{3}\frac{h}{I_2}\right]\left[\frac{L}{I_1} + \frac{2h}{I_2}\right] - \left[\frac{L}{I_1} + \frac{h}{I_2}\right]^2}$$

TABLE A—Fig. 68

Member	L	A	$\frac{L}{A}$	S' (1000#)	$\frac{S'L}{A}$	u_a	$\frac{S'L}{A}u_a$	u_b	$\frac{S'L}{A}u_b$	u_c	$\frac{S'L}{A}u_c$	u_d	$\frac{S'L}{A}u_d$	u_a^2	u_au_b	u_b^2	u_bu_c	u_c^2	u_cu_d	u_d^2	$u_a^2\frac{L}{A}$	$u_b^2\frac{L}{A}$	$u_c^2\frac{L}{A}$	$u_d^2\frac{L}{A}$	$u_au_b\frac{L}{A}$	$u_bu_c\frac{L}{A}$	$u_cu_d\frac{L}{A}$
U_0U_1	240	25.0	10.4	−280.0	−2920.0	−.625	+1825.0							.390							4.06						
U_1U_2	240	40.0	6.0	−480.0	−2880.0			−.625	+1800.0							.390						2.34					
U_2U_3	240	49.0	4.9	−600.0	−2940.0					−.625	+1840.0							.390					1.91				
U_3U_4	240	53.0	4.5	−640.0	−2880.0							−.625	+1800.0							.390				1.75			
L_0L_1	240	23.0	10.4	0.0	0.0	−.625								.390							4.06	2.34					
L_1L_2	240	40.0	6.0	+280.0	+1680.0			−.625	−1050.0							.390											
L_2L_3	240	49.0	4.9	+480.0	+2350.0					−.625	−1470.0							.390					1.91				
L_3L_4	240	53.0	4.5	+600.0	+2700.0							−.625	−1690.0							.390				1.75			
U_0L_0	300	25.0	12.0	−400.0	−4800.0	−.782	+3750.0							.610							7.32						
U_1L_1	300	18.0	16.7	−350.0	−5840.0	.782	+4570.0	−.782	+4570.0					.610	.610	.610					10.20	10.20			10.2		
U_2L_2	300	15.0	20.0	−250.0	−5000.0	...	...	−.782	+3910.0	−.782	+3910.0	...	...	...	...	.610	.610	.610	...	...	...	12.20	12.20	...	...	12.2	
U_3L_3	300	10.0	30.0	−150.0	−4500.0					−.782	+3540.0	−.782	+3540.0					.610	.610	.610			18.30	18.30			18.3
U_4L_4	300	10.0	30.0	−100.0	−3000.0							−.782	+2345.0							.610				18.30			
U_0L_1	384	20.0	19.2	+448.0	+8600.0	+1.00	+8600.0	...	...	...	...	...	...	1.00	...	...	...	...	...	...	19.20						
U_1L_2	384	16.0	24.0	+320.0	+7680.0	...	...	+1.00	+7680.0	...	...	...	...	...	...	1.00	...	...	...	...	...	24.0					
U_2L_3	384	12.0	32.0	+192.0	+6140.0					+1.00	+6140.0							1.00					32.00				
U_3L_4	384	10.0	38.4	+64.0	+2460.0	...	...	...	...	...	...	+1.00	+2460	...	...	...	...	...	...	1.00	19.20	...	...	38.40			
L_0U_1	384	20.0	19.2	...	...	+1.00	...	...	...	...	...	...	...	1.00													
L_1U_2	384	16.0	24.0					+1.00								1.00						24.0					
L_2U_3	384	12.0	32.0							+1.00								1.00					32.00				
L_3U_4	384	10.0	38.4									+1.00								1.00				38.40			
							$\Sigma=+18745.0$ $=E\times\delta'_a$		$\Sigma=+16910.0$ $=E\times\delta'_b$		$\Sigma=+13960.0$ $=E\delta'_c$		$\Sigma=+8455.0$ $=E\delta'_d$								$\Sigma=64.04$ $=E\delta_{aa}$	$\Sigma=75.08$ $=E\delta_{bb}$	$\Sigma=98.32$ $=E\delta_{cc}$	$\Sigma=116.9$ $=E\delta_{dd}$	$\Sigma=10.2$ $=E\delta_{ab}$	$\Sigma=1.22$ $=E\delta_{bc}$	$\Sigma=18.3$ $=E\delta_{cd}$

$=u_du_{d'}\frac{L}{A}$ $\quad = E\delta_{dd'}$

The fundamental equations are

$$\delta_a = 0 = \delta'_a + X_a\delta_{aa} + X_b\delta_{ab}, \quad (a)$$
$$\delta_b = 0 = \delta'_b + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc}, \quad (b)$$
$$\delta_c = 0 = \delta_c' + X_b\delta_{cb} + X_c\delta_{cc} + X_d\delta_{cd}, \quad (c)$$
$$\delta_d = 0 = \delta'_d + X_c\delta_{dc} + X_d\delta_{dd} + X_{d'}\delta_{dd'}$$
$$= \delta'_c + X_c\delta_{dc} + X_d(\delta_{dd} + \delta_{cd}),^{*} \quad (d)$$

* Since $X_d = X_{d'}$, $X_c = X_{c'}$, etc. from symmetry of truss, and $\delta_{dd'} = \delta_{cd}$ from table.

Substituting the δ-values from table (each multiplied by the constant E):

$$64X_a + 10.2X_b = -18745 \quad (a')$$
$$10.2X_a + 75.0X_b + 12.2X_c = -16910 \quad (b')$$
$$12.2X_b + 98.3X_c + 18.3X_d = -13960 \quad (c')$$
$$18.3X_c + 135.2X_d = -8455 \quad (a')$$

Eliminate X_a from (a') and (b') and X_d from (c') and (d'), thus:

$$X_a + .159X_b = -293.0$$
$$X_a + 7.380X_b + 1.2X_c = -1656.0$$
$$7.22X_b + 1.2X_c = -1364.0$$
$$X_c + 6.01X_b = -1136.0$$
$$X_c + .128X_b = -134.7$$
$$5.88X_b = -1001.3$$

$$X_d + .136X_c = -62.5$$
$$X_d + 5.37X_c + .667X_b = -763.0$$
$$5.2X_c + .667X_b = -700.5$$
$$X_c + .128X_b = -134.7$$

$\therefore\ X_b = -\mathbf{170.0}$

$$X_c = -134.7 + .128\times 170 = -\mathbf{112.8}$$
$$X_a = -292.5 + .159\times 170 = -\mathbf{265.0}$$
$$X_d = -62.5 + .136\times 112.8 = -\mathbf{47.2}$$

(To Face Page 114.)

If we let $\dfrac{L}{I_1} = {}_1k,\ \dfrac{h}{I_2} = k_2,\ \dfrac{a}{I_1} = n$ we shall have:

$$X_b = -\frac{Pa}{2h}\,\frac{(n+k_2)(k_1+2k_2)-(n+2k_2)(k_1+k_2)}{(k_1+\frac{2}{3}k_2)(k_1+2k_2)-(k_1+k_2)^2} = -\frac{3Pa}{2h}\,\frac{n-k_1}{2k_1+k_2}$$

$$= \frac{3Pa(L-a)I_2}{2h(2LI_2+hI_1)} = \left(\text{if } \frac{I_1}{I_2}\cdot\frac{h}{L} = k\right)\ \ldots\ P\,\frac{3ab}{2hL(k+2)}$$

The plus sign indicates that $X_b(=H_C)$ acts as indicated in Fig. 67*b*. A similar reduction gives the other redundants as

$$X_a = V_C = \frac{Pa}{L}\left[1+\frac{b}{L_2}\,\frac{L-2a}{6k+1}\right],$$

and

$$X_c = M_C = \frac{Pab}{2L}\cdot\frac{(5k-1)+2\frac{b}{L}(k+2)}{(k+2)(6k+1)}.$$

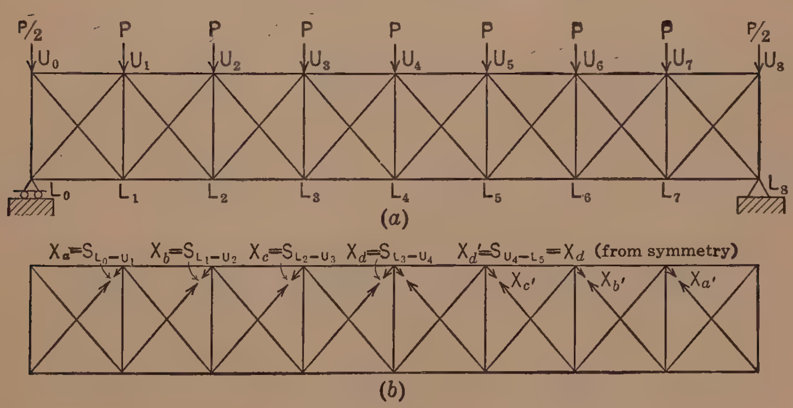


FIG. 68

Problem 2 is an 8-panel quadrangular truss with each panel doubly braced. It is in general 8-fold indeterminate, but for the symmetrical loading of this problem the redundant stresses are identical on either side of the center so that the problem becomes only quadruply indeterminate. Figs. 68*a* and 68*b* show the structure and loading and the equivalent base structure. The solution for the redundant stresses is shown in Table A and the solution of the numerical equations following the table.

When the stresses in the redundant members have been found, the true stress in any member is obtained from the equation,

$$S = S' + X_a u_a + X_b u_b \ldots X_n u_n.$$

SECTION III.—INFLUENCE LINES FOR STATICALLY INDETERMINATE STRUCTURES

38. Simples Cases.—As a general rule influence lines are much more important in the analysis of statically indeterminate structures than in simple structures. In many cases they constitute the only practicable method of determining conditions for maximum and minimum loading. We shall consider a few of the simpler cases.

(*a*) *Two-span Continuous Girder.*

To construct the influence line for R_B in the beam of Fig. 51 we require the equation for R_B due to a unit load acting at any point q. In this case it is convenient to denote δ'_B by δ_{Bq} and from the preceding theory we have at once

$$R_B = -\frac{\delta_{Bq}}{\delta_{BB}} = -\frac{\int_a^c \frac{m_q m_B dx}{EI}}{\int_a^c \frac{m_B{}^2 dx}{EI}}.$$

If then we compute the above numerator for a number of different positions q of the unit load we obtain corresponding points on the influence line for R (obviously δ_{BB} is a constant). This procedure is very tedious, and we shall ordinarily find it advantageous to proceed as follows:

From Maxwell's principle we have $\delta_{Bq} = \delta_{qB}$, where δ_{qB} is the deflection at any arbitrary point q due to a unit load at B. That is to say, if we construct the deflection curve for the beam under a unit load at B, this curve multiplied by $\frac{1}{\delta_{BB}}$ *is the influence line for* R_B. We may conveniently obtain this deflection curve analytically or graphically by the method of elastic weights, using as a load diagram the actual moment diagram for the simple beam AC loaded with unity at B.

(*b*) *Truss on Three Supports.*

If we consider a truss on three supports (Fig. 63) and apply the above general theory, we get

$$R_e = -\frac{\delta_{eq}}{\delta_{ee}} = -\frac{\delta_{qe}}{\delta_{ee}} = -\frac{\sum \frac{u_q u_e L}{AE}}{\sum \frac{u_e{}^2 L}{AE}},$$

where $q = b, c \ldots h$. If we apply a unit load at e to the base-system and construct a Williot displacement diagram for this case, we shall get from this one diagram *all* the values of δ_{qe} and thus all the data for the construction of the influence line for R_e. We may also obtain the deflection line conveniently by means of a simple beam moment diagram

for elastic loads, following essentially the method of Chapter I, Section II, D.

(c) *The Two-hinged Arch Rib.* (Fig. 61.)

For the two-hinged arch rib under a single vertical load unity at any point q,

$$R_H = -\frac{\delta_{Bq}}{\delta_{BB}} = -\frac{\delta_{qB}}{\delta_{BB}} = -\frac{\displaystyle\int\frac{m_q m_b dx}{EI_c}}{\displaystyle\int\frac{m_B{}^2 dx}{EI_c}},$$

if we make the assumptions of Section C, Problem d. Fig. 61c shows influence line for R_H. It should be noted that δ_{Bq} = horizontal displacement at B due to unit load acting vertically at q = vertical displacement at q due to unit load acting horizontally at $B = \delta_{qB}$. The quantities m_q and m_B are respectively the moment at any section due to unity at q acting vertically on the simple curved beam AB, and the moment at any section due to unity applied horizontally at B to the same structure. Here, as in the case of the continuous straight beam, we may construct the influence line for R_H as a moment diagram of a simple beam under certain elastic loads. This is discussed in the Chapter on Arches.

(d) *Two-hinged Braced Arch.*

The above formula for R_H holds if we substitute the truss-deflection expression instead of the corresponding form for beams. Thus

$$R_H = -\frac{\delta_{gq}}{\delta_{gg}} = -\frac{\displaystyle\sum\frac{u_q u_g L}{AE}}{\displaystyle\sum\frac{u_g{}^2 L}{AE}} = -\frac{\delta_{qg}}{\delta_{gg}}.$$

Here again we may obtain all the values of δ_{qg} from a single Williot diagram. We apply a unit horizontal force at g, no other loads acting, and draw the displacement diagram. From this we obtain the vertical deflection of each joint B, C . . . F, which by Maxwell's principle is numerically equal to the horizontal deflection at g, due to a unit vertical load at q, and hence is the desired quantity.

39. General Method for Multiply Redundant Structures.

Let us take for example a triply indeterminate structure for which we have the equations:

$$X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} = -\delta_{aq}$$
$$X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} = -\delta_{bq}$$
$$X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc} = -\delta_{cq}.$$

Solving these equations for the X's, and noting that $\delta_{ab} = \delta_{ba}$, etc., we get

$$X_a = -\frac{\delta_{aq}(\delta_{bb}\delta_{cc} - \delta_{bc}^2) + \delta_{bq}(\delta_{ac}\delta_{bc} - \delta_{ab}\delta_{cc}) + \delta_{cq}(\delta_{ab}\delta_{bc} - \delta_{ac}\delta_{bb})}{\Delta},$$

if

$$\Delta = \begin{vmatrix} \delta_{aa} & \delta_{ab} & \delta_{ac} \\ \delta_{ba} & \delta_{bb} & \delta_{bc} \\ \delta_{ca} & \delta_{cb} & \delta_{cc} \end{vmatrix};$$

and two similar equations for X_b and X_c.

We may write the above equations

$$X_a = -k_{aa}\delta_{qa} - k_{ab}\delta_{qb} - k_{ac}\delta_{qc}, \quad \ldots \ldots \quad (30)$$

where

$$k_{aa} = \frac{\delta_{bb}\delta_{cc} - \delta_{bc}^2}{\Delta}; \text{ etc.}$$

and two similar for X_b and X_c.

Now, δ_{qa}, δ_{qb}, etc., are respectively the deflections at any point q in the base-system, due to unit loadings at a, b, and c. Therefore,

$$k_{aa}\delta_{qa},\ k_{ab}\delta_{qb},\ k_{ac}\delta_{qc},$$

are the deflections at q due to loadings at a, b, c, *numerically* equivalent to k_{aa}, k_{ab}, etc. We have thus reduced the problem of constructing the influence line for any of the statically indeterminate quantities X to the problem of constructing the deflection line for the statically determined base-system—the simple structure resulting from the removal of all redundant bars or supports—under certain elastic loads k applied at the points of redundancy. This deflection line of the simple structure is ordinarily most easily obtained by the method of elastic weights, or, if a truss, by the Williot diagram.

As an example we may take the four span continuous girder of Fig. 66. We apply to the base-system (simple beam AB) the forces

$$X_a = -k_{aa},\ X_b = -k_{ab},\ X_c = -k_{ac}.$$

This loading will give the moment diagram of Fig. 66*d*. To obtain the elastic curve we apply the moment diagram as a load curve and thus obtain the curve of Fig. 66*e*. If we determine the scale by the fact

that $y_a = 1$ (since for a unit load at a, X_a must obviously equal unity) then from the preceding theory, $A''a''b''c''B''$ is the true influence line for the reaction X_a. The method is general and may be applied to other problems than the straight continuous girder or truss. Fig. 69 shows a continuous arch with the influence line for the horizontal thrust constructed by this method, but a full treatment of the subject is beyond the scope of this treatise.*

40. Mechanical Solution.—A most ingenious application † of the preceding principles, with certain modifications, has led to a mechanical solution of statically indeterminate structures, apparently applicable to all types, whatever the degree of indetermination, and which promises to be of great practical importance. Only the outline of the method can be presented here, and we may do this by showing the application to the continuous girder of Fig. 66.

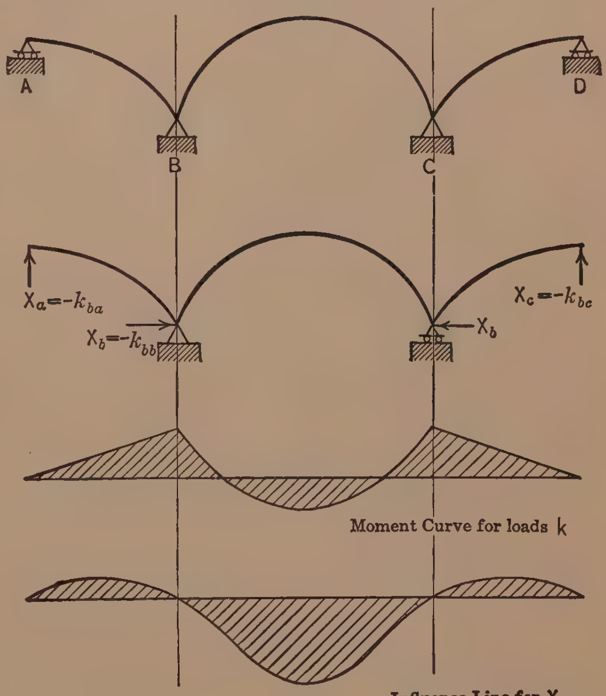


FIG. 69

In this method the fundamental structure is not the simple structure with all redundants removed; it is the structure obtained by the removal of the redundant whose value we seek, and no other. Let us suppose

* For further treatment of the the subject, the reader is referred to H. Mueller-Breslau "Die Neueren Methoden der Festigkeitslehre," which is largely the source of the above discussion (pp. 195–228, Edition of 1913).

† Due to Professor George E. Beggs of Princeton University, to whom the authors are indebted for interesting information regarding results that have been obtained in solving indeterminate problems mechanically. See article by Professor Beggs in Proc. Am. Conc. Institute, Vol. XVIII, pp. 58–82.

that in the above girder we want to derive the influence line for X_a. We remove the support at a and consider the continuous girder $AbcB$. If we denote the deflection of any point of the structure by Δ (to distinguish from the previous case where the base-system was the simple beam AB), a little reflection will serve to show that we must have

$$X_a = R_a = -\frac{\Delta_{aq}}{\Delta_{aa}} = -\frac{\Delta_{qa}}{\Delta_{aa}},$$

where in general Δ_{qr} = deflection at q due to unity at r *in the continuous girder* $AbcB$.

But, since Δ_{aa} is a constant, this means that if we have the deflection curve for the continuous beam $AbcB$ for a unit load at a, this must be *to some scale*, the influence line for R_a. For the ordinary course of analytical calculation, to be sure, such a procedure is futile; we should have to carry out a solution of the statically indeterminate structure $AbcB$ before the deflection curve could be found. But if we lay out, on a drawing board or otherwise, the spans Aa, ab, bc, cB to scale and place on the supports A, b, c, and B a flexible bar of homogeneous material (so-called "spline"), having I proportional to that of the actual girder, we then have a simple mechanical means of obtaining the desired deflection line. Hinging the spline at A, b, c, and B, we displace the point a an amount $y_a = 1$. Then the ordinate at any other point, q, measured from the base line $AbcB$ to the neutral line of the spline, is equal to the reaction at a due to unity at q. It is obvious that the spline will take a curve identical in form with $AbcB$ in Fig. 66*e*.

In general, for any statically indeterminate structure, if we effect, on a model of the structure, a unit displacement at the point of application of the redundant force (unit *angular* displacement if the redundant is a couple) and measure the displacement in a given direction of any other point, this will equal the value of the redundant force for a unit load at the point acting in the given direction. Good results have been obtained by the use of relatively simple and easily constructed cardboard models.

SECTION IV.—THE METHOD OF LEAST WORK

41. General Theory.—In Section I, Chapter I, we developed the expression for deflection as the partial derivative of the internal work of deformation,

$$\delta_r = \frac{\partial W}{\partial P_r}.$$

If now we have a beam or truss with a single redundant support, which we replace as in the preceding cases by an unknown force X, we must have (if the support is unyielding),

$$\delta = \frac{\partial W}{\partial X} = 0,$$

which gives the required equation for X.

If we have a reaction in the form of a restraining moment (as in a fixed-ended beam), the same equation holds if we understand δ to be a general term for displacement, including angular as well as linear movements. X is then the applied external couple statically equivalent to the restraining moment.

In the case of a frame with a redundant bar, if, as usual, we sever the bar and apply a force-pair X (equivalent to the true stress in the bar) to the cut faces, and if we call the relative displacement of these faces δ, the preceding equation is still valid.

If there are several statically indeterminate quantities, we shall have, since W is in general a function of all these quantities,

$$\delta_a = \frac{\partial W}{\partial X_a} = 0; \quad \delta_b = \frac{\partial W}{\partial X_b} = 0; \quad \ldots \delta_n = \frac{\partial W}{\partial X_n} = 0. \quad . \quad (31)$$

We thus have an equation of condition for every redundant and the method is perfectly general.

If $W = f(X_a, X_b \ldots X_n; \quad P_a, P_b \ldots P_n)$ where the loads P are to be regarded as constants throughout the investigation, the values of the X's determined by the conditions

$$\frac{\partial W}{\partial X_a} = 0, \quad \frac{\partial W}{\partial X_b} = 0, \text{ etc.}$$

are the values that cause W to take either a maximum or a minimum value. Physical considerations indicate that it cannot be a maximum. For in the case of, say, a continuous girder or truss, if we apply to the base-system forces X having the same sense as the specified loads, it is clear that W increases uniformly as the X's increase, and no true maximum is possible. Similar reasoning applies to other indeterminate

structures. We appear justified therefore in assuming that values of X determined as above render W a minimum.*

We thus arrive at this important generalization: In every case of statical indetermination where an indefinite number of different values of the redundant forces X will satisfy all statical requirements, the true values are those which render the total internal work of deformation a minimum.

This law generally goes by the name of the "principle of least work." It is often urged as peremptory proof of the principle that it must follow from the "economy of nature" that all natural operations take place with a minimum expenditure of energy.

It has been held that the principle is traceable to the "principle of least action" which has played so important a part in the development of mathematical physics. As a principle useful in the analysis of stresses in structures, it appears to be due to Menabrea (1858). But it was discovered independently by Castigliano (1875) and its application greatly extended, whence it is generally known as Castigliano's second theorem. Fränkel also arrived at the principle independently (1882).

42. Method of Application.—To illustrate in a general way the application of the method of least work, let us take the case of a con-

* For a single redundant the mathematical proof follows readily:

$$W = \frac{1}{2}\sum\int\frac{M^2ds}{EI} + \frac{1}{2}\sum\int\frac{N^2ds}{AE};$$

$$\frac{\partial W}{\partial X} = \sum\int\frac{Mds}{EI}\cdot\frac{\partial M}{\partial X} + \sum\int\frac{Nds}{AE}\frac{\partial N}{\partial X};$$

$$\frac{\partial^2 W}{\partial X^2} = \sum\left[\int\frac{Mds}{EI}\cdot\frac{\partial^2 M}{\partial X^2} + \int\frac{ds}{I}\left(\frac{\partial M}{\partial X}\right)^2\right]$$

$$+\sum\left[\frac{Nds}{AE}\cdot\frac{\partial^2 N}{\partial X^2} + \int\frac{ds}{AE}\left(\frac{\partial N}{\partial X}\right)^2\right].$$

But if M and N are linear functions of X,

$$\frac{\partial^2 M}{\partial X^2} = \frac{\partial^2 N}{\partial X^2} = 0,$$

whence

$$\frac{\partial^2 W}{\partial X^2} = \sum\int\frac{ds}{EI}\cdot\left(\frac{\partial M}{\partial X}\right)^2 + \sum\int\frac{ds}{AE}\cdot\left(\frac{\partial N}{\partial X}\right)^2,$$

an essentially positive quantity.

The general case of maxima and minima of a function of several variables involves other considerations, and a really rigorous investigation of the question from this standpoint is hardly in place here. It is believed that the physical argument is quite convincing.

tinuous girder of three spans. We replace the effect of the intermediate supports by the undetermined external forces X_a and X_b as in previous cases. We have,

$$W = \frac{1}{2}\int_A^B \frac{M^2dx}{EI},$$

where A and B are the end points of the entire girder system. Then the equations of condition are

$$\frac{\partial W}{\partial X_a} = 0 = \int_A^B \frac{Mdx}{EI}\cdot\frac{\partial M}{\partial X_a}; \quad \frac{\partial W}{\partial X_b} = 0 = \int_A^B \frac{Mdx}{EI}\cdot\frac{\partial M}{\partial X_b}.$$

We may write

$$M = M' + M_a + M_b,$$

where M' is the simple beam moment in the span AB, just as we have hitherto used it, and M_a and M_b are respectively the moments at any point in the simple beam AB due to forces X_a and X_b applied singly to the points a and b, no other forces acting. We then have

$$\frac{\partial W}{\partial X_a} = \int_A^B \frac{M'dx}{EI}\cdot\frac{\partial M_a}{\partial X_a} + \int_A^B \frac{M_adx}{EI}\cdot\frac{\partial M_a}{\partial X_a} + \int_A^B \frac{M_bdx}{EI}\cdot\frac{\partial M_b}{\partial X_a} = 0. \quad (32)$$

Changing the subscript from a to b we get a similar equation for $\frac{\partial W}{\partial X_b}$. M', M_a, M_b, $\frac{\partial M_a}{\partial X_a}$, $\frac{\partial M_b}{\partial X_b}$, are all easily obtained, and the integrals readily evaluated. There result two equations in X_a and X_b and certain constants from which we find the values of the former quantities.

The example of Fig. 70 will illustrate the method of procedure.

Recalling that

$$M_a = X_a m_a, \; M_b = X_b m_b,$$

and therefore

$$\frac{\partial M_a}{\partial X_a} = m_a, \; \frac{\partial M_b}{\partial X_b} = m_b,$$

Equation (32) is transformed into

$$\frac{\partial W}{\partial X_a} = \delta_a = 0 = \int_A^B \frac{M'dx}{EI}\cdot m_a + X_a\int_A^B \frac{m_a{}^2dx}{EI} + X_b\int \frac{m_am_bdx}{EI}$$

$$= \delta'_a + X_a\delta_{aa} + X_b\delta_{ab},$$

which the student will readily identify with the general equation (29).

43. Summary.—The method of least work has played a very important part in the development of the theory of structures, and it is still widely used. It is a general method, coordinate with the Maxwell-

Mohr method; in practically all cases of importance to the structural engineer a problem which can be solved by one method can be solved by the other. Opinions differ as to the relative advantages; the authors of this book have felt that on the whole the balance is in favor of the Maxwell-Mohr method. and hence have adopted it as the fundamental

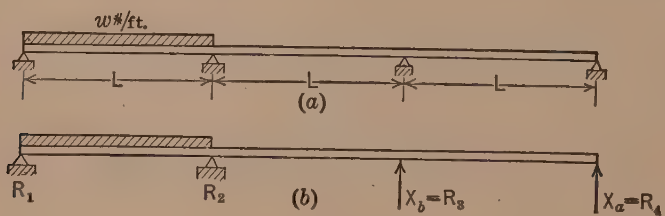


FIG. 70.

Treating the two right-hand supports as redundant we have for the internal work of deformation:

$$W = \sum\int_0^L \frac{M^2dx}{2EI} = \frac{1}{2EI}\int_0^L \left(R_1x^2 - R_1wx^3 + \frac{w^2x^4}{4}\right) dx$$

$$+ \frac{1}{2EI}\int_0^L [(R_3 + R_4)x^2 + R_4L(R_3 + R_4)x + R_4^2L^2]dx + \frac{1}{2EI}\int_0^L R_4^2x^2dx$$

$$= \frac{L^3}{2EI}\left[\frac{R_1^2}{3} - R_1\frac{wL}{4} + \frac{w^2L^2}{20} + \frac{R^2{}_4}{3} + \frac{R^2{}_3 + 2R_3R_4 + R_4^2}{3} + R_3R_4 + 2R_4^2\right] \quad (A)$$

$$\frac{\partial W}{\partial R_3} = \frac{L^3}{2EI}\left[\frac{2}{3}R_1\frac{\partial R_1}{\partial R_3} - \frac{wL}{4}\frac{\partial R_1}{\partial R_3} + \frac{2}{3}R_3 + \frac{2}{3}R_4 + R_4\right] = 0, \quad . \; . \; . \; . \; . \; . \quad (B)$$

and

$$\frac{\partial W}{\partial R_4} = \frac{L^3}{2EI}\left[\frac{2}{3}R_1\frac{\partial R_1}{\partial R_4} - \frac{wL}{4}\cdot\frac{\partial R_1}{\partial R_4} + \frac{2}{3}R_4 + \frac{2}{3}R_3 + \frac{2}{3}R_4 + R_3 + 4R_4\right] = 0. \quad (C)$$

From statics we have . . . $R_1 = \frac{wL}{2} + R_3 + 2R_4$, whence $\frac{\partial R_1}{\partial R_3} = 1$ and $\frac{\partial R_1}{\partial R_4} = 2$, ∴ substituting in (*B*) and (*C*) and collecting terms we get:

$\frac{wL}{12} + \frac{4}{3}R_3 + R_4 = 0$, and $\frac{wL}{6} + 3R_3 + 8R_4 = 0$, whence $R_4 = \frac{wL}{60}$, $R_3 = -\frac{wL}{10}$; and $R_1 = \frac{13}{30}wL$. $R_2 = wL - [R_1 + R_3 + R_4] = wL - \frac{21}{60}wL = \frac{39}{60}wL = \frac{13}{20}wL$.

method for the general treatment of statically indeterminate problems. But it should be said that in spite of the differences in the fundamental conceptions of the two methods, the parallelism in the actual detail of applications to problems is so close that there is very little to choose between them on that score. The historical importance of the method of least work and the fact that such wide use is still made of it in con-

temporary literature has made it seem desirable to explain its fundamental character, though, for the reasons just stated, little further use will be made of it.

The whole system of analytical treatment based on the internal work of deformation is sometimes referred to as the "method of least work." Though the distinction may not be of great practical importance, for the sake of clear thinking it is well to note that such usage is incorrect.

SECTION V.—TEMPERATURE AND OTHER NON-ELASTIC EFFECTS

44. Modification of Preceding Formulas.—In the preceding derivation of formulas for the redundants in a statically indeterminate structure we have omitted from consideration the effect of temperature, of yielding supports, slip of riveted joints and other similar effects.

If in the beam ABC the support B sinks a small distance Δ_B below the level AC, we can no longer write

$$\delta_B = 0 = \delta'_B + R_B\delta_{1B},$$

but we must write

$$\delta_B = \Delta_B = \delta'_B + R_B\delta_{1B}.$$

$$\therefore \quad R_B = -\frac{\delta'_B - \Delta_B}{\delta_{1B}}.$$

Again let us suppose that when the support B is removed and the loads are applied to the base structure AC, an unequal distribution of temperature takes place so that there results a displacement from this cause which we call Δ_{tB}. Then clearly

$$R_B = -\frac{\delta'_B \pm \Delta_{tB}}{\delta_{1B}}.$$

So in the truss of Fig. 52 if when the redundant member is cut and the loads are applied there are also temperature changes in the different members, then in general the total displacement of the cut faces will be $\delta' \pm \Delta_t$ and

$$S_r = -\frac{\delta' \pm \Delta_t}{\delta_1}.$$

A similar provision may be made for other non-elastic distortions.

It is clear that we may express the effect of temperature or other similar change on the redundant independently of the effect of the loads by placing $\delta' = 0$ whence

$$X = \frac{\pm \Delta_t}{\delta_1}.$$

Temperature effects will be treated further under the special problems of the later chapters of the book.

SECTION VI.—GENERAL SUMMARY

45. The following summary may aid the student in gaining a clearer view of the subject as a whole.

(*a*) The first step in attacking a statically indeterminate problem by the general method is to decide (if, as is usual, there are alternatives) on the base-system. The second step is to replace the redundants by statically equivalent external forces X, acting on the base-structure. The third step is to write for the above structure the displacement equations for the points of application * of X. These displacements must be known or the problem is incapable of solution. If the redundant is a single superfluous bar, we cut it at the end and express the relative displacement of its faces, which we know, if the force pair X is equal to the true stress in the bar, must be zero. If the redundant is a single superfluous reaction or restraining moment, we know that its resultant displacement must be zero, assuming the ordinary case of rigid support. (Temperature, settling of supports, slip of joints, etc., are generally provided for separately. See Section V.) The deflection equations take the general form (29).

$$\delta_r = 0 = \delta'_r + X_a\delta_{ra} \dots X_r\delta_{rr} \dots X_n\delta_{rn},$$

which equation merely states that the final deflection of the point r is the deflection which the given loading, acting alone, would produce in the simple structure, combined with the deflection which the redundant forces, acting alone, would produce in the same structure.

It is important to note that thus far the equation does not require the principle of work for its establishment; it depends only on the principle of the proportionality of deflection to load (which establishes that the deflection due to X_r equals $X_r\delta_{rr}$) and the principle of superposition, which states that the effect of a set of forces applied simultaneously to a structure is equal to the sum of the effects of the forces applied separately.

It is only when we attempt to evaluate the quantities δ that we must have recourse to one of the several methods developed in Chapter I. The form of the equation illustrates very clearly the fact that the problem of determining the redundants in a statically indeterminate structure is essentially but a problem in deflections.

(*b*) A different philosophical aspect of the problem is brought out by the principle of least work, but for the problems treated in this book and for most structural problems, the practical difference is slight. We

* See p. 110 for interpretation of the term "*point* of application."

set up an equation for the total internal work of deformation, and differentiate successively with respect to the redundants X, and, to determine the X's so that the total work is a minimum, we must have these derivatives equal to zero. This gives as many independent equations as there are redundants. But, in order to set up the equation of internal work, it is necessary to treat the structure as a statically determined base system acted upon by the given loads and by the forces X. Since, by Castigliano's first theorem $\frac{\partial W}{\partial X_r} = \delta_r$ the operation of treating W for a minimum is essentially the same procedure as that followed in (*a*) above, and we have seen (Section IV) that the resulting equations are identical.

(*c*) Either the Maxwell-Mohr principle of the dummy unit loading or Castigliano's principle of least work results in a perfectly general method of attack directly applicable to any statically indeterminate problem, in so far as the structure can be regarded as assemblage of bars (including the single beam as a special case), straight or slightly curved, and subjected to axial stress or flexure, or both. The advantage of a comprehensive general method for the treatment of such problems, for purposes of demonstration, of unification of the theory and as a check on special methods, requires no comment. But it is not to be expected that the method which has the widest application shall always or indeed usually prove the simplest. Numerous artifices may be used in particular cases to simplify the detail work of application, and in many cases it will be found advantageous to proceed by special methods different from those outlined in the present chapter. In the following chapter we shall consider the subject of special solutions in some detail. It may be well to remark that however markedly some of these modes of attack may vary from the general methods developed in this chapter, they are all fundamentally in harmony with and usually derivable from this method.

CHAPTER III

SPECIAL METHODS OF ATTACK

46. Preliminary.—It has been noted in the preceding chapter that the general method there developed is directly applicable to *any* statically indeterminate problem. No simpler method, possessing equal generality, is known. But for special types of problems, modifications of the general method or independent methods may be devised which are much shorter and readier of application. Sometimes these methods are applicable to very wide groups of problems and are of the highest practical importance. It is the purpose of this chapter to discuss some of the leading methods by which the analysis of statically indeterminate stresses may be simplified in special cases.*

47. The general method of the last chapter presents two main difficulties. First, the evaluation of the quantities δ (Eqs. 29) by the work equations is likely to be quite laborious, and second, in case of multiply indeterminate cases, the solution of the group of simultaneous equations, equal in number to the statically undetermined quantities, is a tedious process and one from which it is difficult to eliminate numerical errors. We may say in general that the chief merit of most (though not all) special methods of attack lies in a simplification along one or both these lines.

48. We shall consider briefly

I. The use of the principle of moment areas and elastic weights to evaluate the deflections in beam problems.

II. The choice of the base-system so as to reduce the number of terms entering the equations for the statically undetermined quantities. (Three moment theorem, etc.)

III. The direct application of the moment area method.

IV. The slope-deflection method.

* The special methods of solution for indeterminate problems are so many and varied that no adequate account of the subject can be given here. This chapter will merely attempt to indicate the general lines along which the simplifications are usually made, and present in detail a few of the more important methods which have proved especially advantageous as practical working methods.

SECTION I.—MOMENT AREAS AND ELASTIC WEIGHTS APPLIED TO EVALUATION OF DEFLECTIONS

49. General.—It was stated in Chapter II that Eqs. 29

$$\delta_a = 0 = \delta'_a + X_a\delta_{aa} \ldots X_n\delta_{an}$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\delta_n = 0 = \delta'_n + X_a\delta_{na} \ldots X_n\delta_{nn},$$

do not require the principle of work for their establishment. We are at liberty to determine the deflections in any manner we please. We have seen in Chapter I that the method of work appears to be the most comprehensive single method, but we have also learned that for *beam* problems the method of elastic weights and moment areas is usually much simpler and more expeditious. We shall now apply this method to some of the problems solved in Chapter II by the method of work.

50. Examples.

1. If we take the beam of example (*a*), p. 95 (Fig. 71), we may write at once

$$R_B = -\frac{\delta'_B}{\delta_{1B}} = -\frac{Pa\cdot\frac{a^2}{2}\cdot\left(b+\frac{2}{3}a\right)}{\frac{L^2}{2}\cdot\frac{2}{3}L} = -\frac{\frac{Pa^2}{6}(3b+2a)}{\frac{L^3}{3}} = -\frac{Pa^2}{2}\cdot\frac{3L-a}{L^3}.$$

2. Consider the beam of Fig. 56, p. 96, to find R_B (Fig. 72), using data of Table I, Chapter I on parabolic moment diagram,

$$R_B = -\frac{\delta'_B}{\delta_{1B}} = -\frac{\frac{1}{3}\cdot\frac{wa^3}{2}\left(L-a+\frac{3}{4}a\right)}{\frac{L^3}{3}} = -\frac{w}{8}\left(\frac{a}{L}\right)^3(4L-a).$$

3. Consider same beam as above, to find M_A as the redundant (Fig. 73).

$$M_A = -\frac{\alpha'_A}{\alpha_{1A}} = -\frac{\left(\frac{L-\frac{a}{2}}{L}\right)\cdot\frac{2}{3}\cdot\frac{wa^3}{8}+\frac{wa^2b}{2L}\cdot\frac{L+b}{3L}}{\frac{L}{2}\cdot\frac{2}{3}\frac{L}{L}}$$

$$= \frac{wa^2}{8}\left[\frac{a(2L-a)+2(L-a)(2L-a)}{L^2}\right] = \frac{wa^2}{8}\left(\frac{2L-a}{L}\right)^2.$$

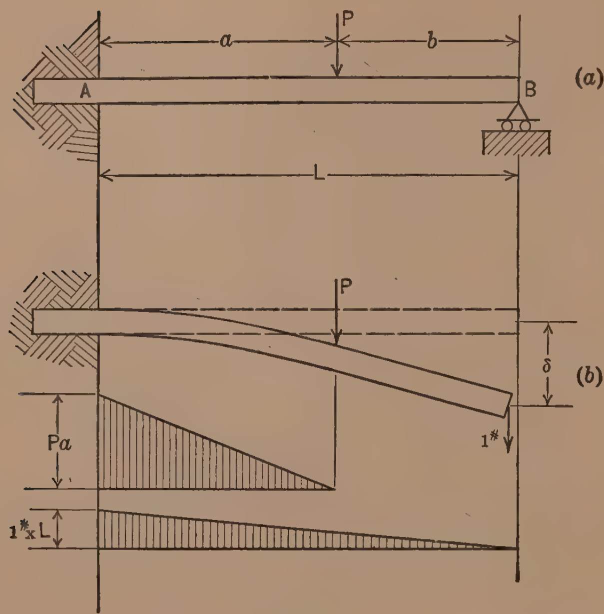


FIG. 71

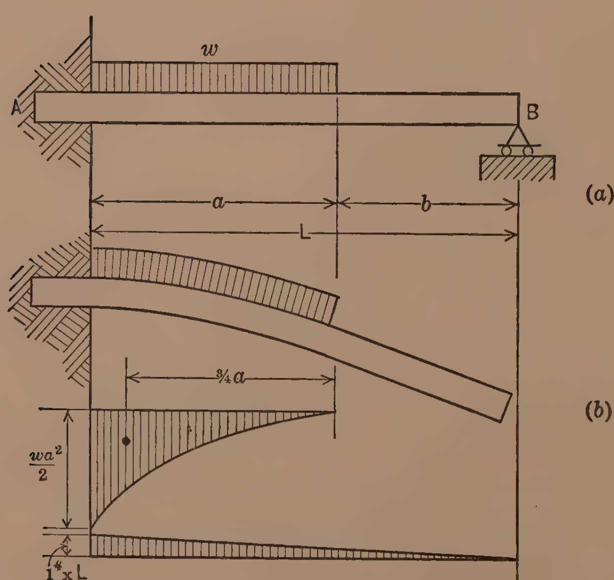


FIG. 72

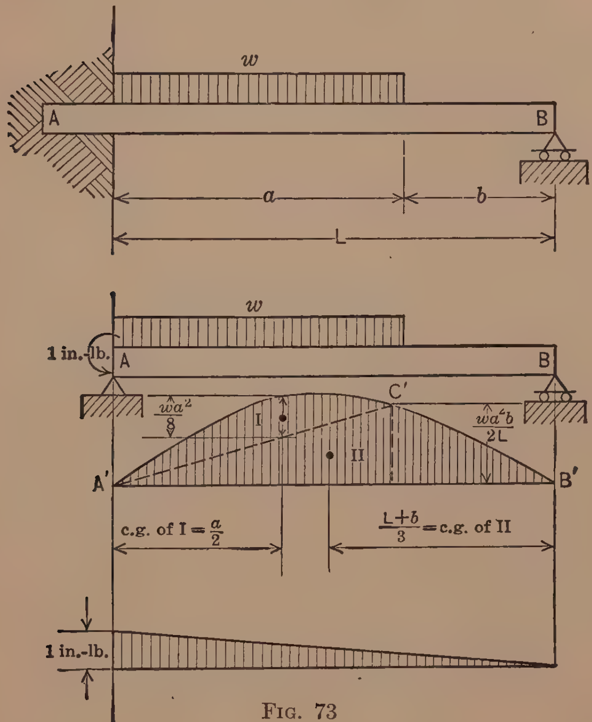

w
A
B
a
b
L
w
1 in.-lb.
A
B
$\frac{wa^2}{8}$
C'
$\frac{wa^2b}{2L}$
I
II
A'
B'
c.g. of I = $\frac{a}{2}$
$\frac{L+b}{3}$ = c.g. of II
1 in.-lb.

FIG. 73

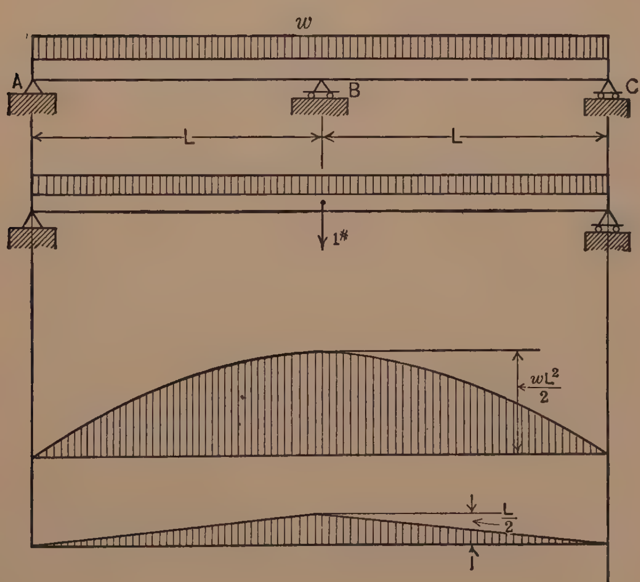

w
A
B
C
L
L
1#
$\frac{wL^2}{2}$
$\frac{L}{2}$

FIG. 74

4. If we wish to find the center reaction in a two span continuous girder, load uniform and spans equal, we have (see Fig. 74),

$$R_B = -\frac{\delta'_B}{\delta_{1B}} = -\frac{\frac{2}{3}\cdot\frac{wL^4}{2} - \frac{2}{3}\frac{wL^3}{2}\cdot\frac{3}{8}L}{\frac{L^3}{4} - \frac{1}{3}\frac{L^3}{4}} = -\frac{5}{4}wL.$$

SECTION II.—SPECIAL SELECTION OF BASIC STRUCTURE

51. General.—For a multiply indeterminate structure the directness and simplicity of the solution is importantly affected in many cases by the choice of the statically equivalent substitute structure which is used as a base-system. It is impossible to give general rules to cover all cases, but we may note that ordinarily it is advantageous to break up the structure, if possible, into more or less independent parts such that the effect of the loads and the redundant forces X do not extend

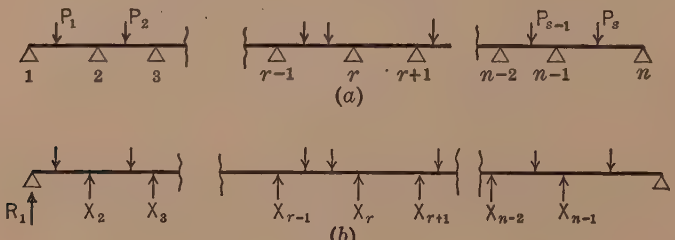


FIG. 75

over the whole system. This means that fewer terms will appear in Eqs. 29. The following examples will make this point clear, and they will also show that the application of this principle leads to methods of attack possessing considerable generality.

52. Application to a Continuous Girder with n Supports.—(Fig. 75*a* and 75*b*.)

First type of base-system.—We may select as the base structure the simple beam (1) — (n), Fig. 75*b*. The redundants here are the $n - 1$ intermediate *reactions*. Then we must have (from Eqs. 29) e.g.

$$\delta_2 = 0 = \delta'_2 + X_2\delta_{22} + X_3\delta_{23} \ldots X_r\delta_{2r} \ldots X_{n-1}\delta_{2(n-1)},$$

and $n - 1$ similar equations. δ'_2 is the downward deflection of the support point (2) due to loads P acting on the simple span (1) – (n); the remaining terms in the right hand member of the equation represent,

collectively, the upward deflection of the same point due to the forces X (equivalent to true reactions) acting on the simple span (1) - (n). δ_{22} is the deflection at (2) due to $X_2 = 1$ no other forces acting; δ_{2r} is the deflection at (2) due to $X_r = 1$, no other force acting, etc. It is obvious that the calculations for each δ will extend over the entire beam, and that each of the $n - 1$ equations will contain a full complement of terms.

53. *Second type of base-system.* Let us consider next a differently selected base-system (Fig. 76). Here the structure is replaced by a series of simple beam spans. The redundants are the *moments* at the $n - 1$ intermediate supports. If these moment-pairs

$$X_2 \dots X_r \dots X_{n-1},$$

as shown in the figure are of such magnitude as to maintain a common

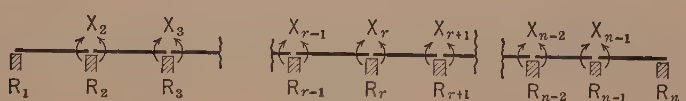


FIG. 76

tangent at points (2) -- (r) etc., then the structure of Fig. 76 is the statical equivalent of that shown in Fig. 75*a*.

Eqs. 29 become

$$\left.\begin{aligned} \delta_2 &= 0 = \delta'_2 + X_2\delta_{22} + X_3\delta_{23}, \\ \delta_3 &= 0 = \delta'_3 + X_2\delta_{32} + X_3\delta_{33} + X_4\delta_{34} \\ \delta_r &= 0 = \delta'_r + X_{r-1}\delta_{r(r-1)} + X_r\delta_{rr} + X_{r+1}\delta_{r(r+1)} \end{aligned}\right\} \quad . \quad . \quad (33)$$

δ'_r is the relative *angular* displacement of the end tangents at r in the simple spans (r) - (r-1) and (r) - (r+1), due to the loads P. δ_{rr} is the relative angular displacement of the end tangents at r in the adjoining simple beams when the structure is loaded with $X_r = 1$, all other loads removed; $\delta_{r(r-1)}$ is the relative angular displacement at r due to $X_{r-1} = 1$ acting alone on the structure. If r is any support point, it is evident that δ'_n will be affected only by the loads in the immediately adjoining spans, and that none of the X's other than X_{r-1}, X_r, and X_{r+1} can affect the angular displacement at (r)—hence all the δ's but three will in general vanish in each equation.

54. Theorem of Three Moments.—To carry the application a little further, there is shown in Fig. 77 two consecutive spans of a system of continuous beams. If M_r in general is the bending moment

over any support r, and θ_r the relative angular change, at the support r, between the end tangents of the adjacent simple beams, then

$$X_r = M_r \quad \text{and} \quad \delta_r = \theta_r.$$

If we assume the possibility of a relative displacement of supports,

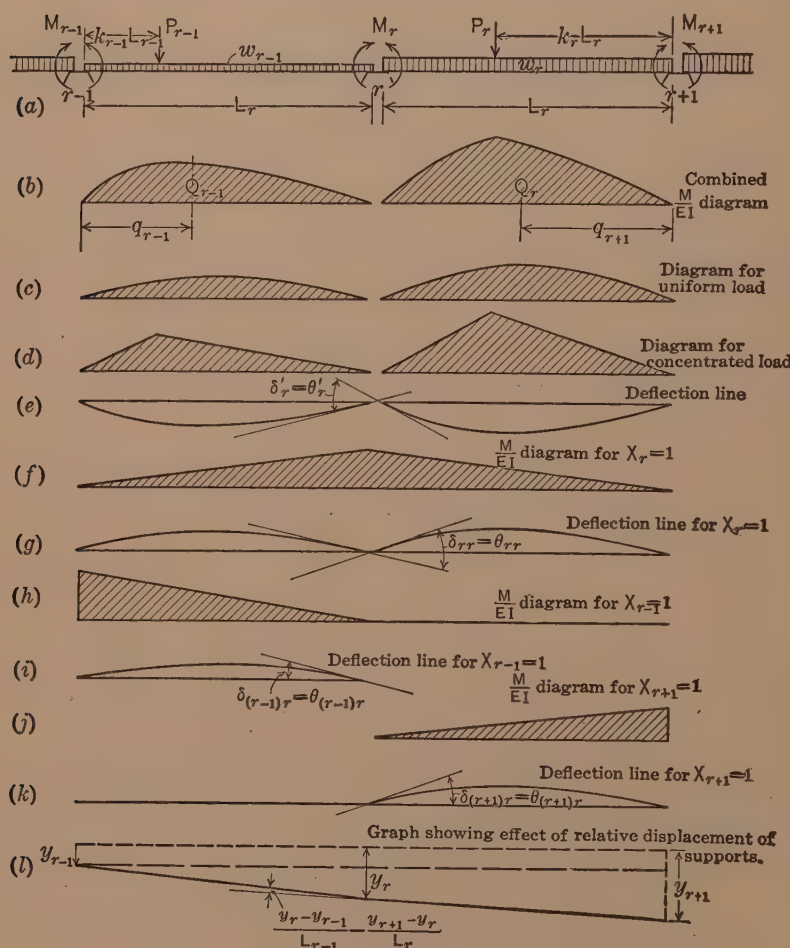


FIG. 77

then the total relative angular displacement between the end tangents at r of the two simple beams $(r)-(r-1)$ and $(r)-(r+1)$ will be:

— Displacement due to yielding supports + displacement due to given loadings P and w + displacement due to the moment-pairs X at

the supports (which are introduced as external forces acting on the series of simple beams) = 0, or (from Fig. 77).

$$-\left(\frac{y_r - y_{r-1}}{L_{r-1}} - \frac{y_{r+1} - y_r}{L_r}\right) + \delta'_r + X_{r-1}\delta_{r(r-1)}$$
$$+ X_r\delta_{rr} + X_{r+1}\delta_{r(r+1)} = 0. \quad . \quad . \quad . \quad (34)$$

or

$$-\left(\frac{y_r - y_{r-1}}{L_{r-1}} - \frac{y_{r+1} - y_r}{L_r}\right) + \frac{M_{r-1}L_{r-1}}{6EI_{r-1}} + M_r\left(\frac{L_{r-1}}{3EI_{r-1}} + \frac{L_r}{3EI_r}\right)$$
$$+ \frac{M_{r+1}L_r}{6EI_r} = -\frac{Q_{r-1}q_{r-1}}{L_{r-1}} - \frac{Q_r q_{r+1}}{L_r} \quad . \quad (34a)$$

Here Q_r = area of $\frac{M}{EI}$ diagram for the span L_r and q_{r+1} the arm of the centroid referred to $r + 1$, q_{r-1} = arm of centroid of Q_{r-1} referred to $r - 1$. It will be recalled from Chapter I that the angular displacement θ at the end of a simple beam due to any loading is the shear at that end for a load equal to the $\frac{M}{EI}$ diagram for the loading considered. All the δ's of eq. 34 are evaluated by means of this simple principle. It is customary to split up the moment diagram Q into the portions due to uniform load and those due to concentrated loads. (See (c) and (d) of Fig. 77.)

The simple beam deflection lines are shown in Fig. 77 (e), and (f) to (k) show the moment diagrams and deflection lines for the redundant moments acting independently. (l) shows the effect of the relative end displacements.

We find that the angular change θ at the end of a simple beam due to a uniformly distributed load is $\frac{wL^3}{24EI}$ and to a single concentrated load distant $L(1 - k)$ from the support, $\theta = \frac{PL^2}{EI}(k^2 - k^3)$. Elementary simplifications then give us eq. 34a in the following form (assuming more than one concentrated load)

$$\frac{M_{r-1}L_{r-1}}{I_{r-1}} + 2M_r\left(\frac{L_{r-1}}{I_{r-1}} + \frac{L_r}{I_r}\right) + M_{r+1}\frac{L_{r+1}}{I_{r+1}}$$
$$= \left(\frac{y_r - y_{r-1}}{L_{r-1}} - \frac{y_{r+1} - y_r}{L_r}\right)6E - \frac{w_{r-1}L^3{}_{r-1}}{4I_{r-1}} - \frac{w_r L_r^3}{4I_r}$$
$$-\sum\frac{PL^2{}_{r-1}(k^2 - k^3)}{I_{r-1}} - \sum\frac{PL^2{}_r(k^2 - k^3)}{I_r}. \quad . \quad . \quad (34b)$$

(The subscripts are omitted from the P's and k's since no misunderstanding is likely to occur from this source). If the I's are constant, we get

$$M_{r-1}L_{r-1} + 2M_r(L_{r-1} - L_r) + M_{r+1}L_{r+1}$$
$$= 6EI\left(\frac{y_r - y_{r-1}}{L_{r-1}} - \frac{y_{r+1} - y_r}{L_r}\right) - \frac{w_{r-1}L^3{}_{r-1}}{4} - \frac{w_rL^3{}_r}{4}$$
$$- \Sigma PL^2{}_{r-1}(k^2 - k^3) - \Sigma PL^2{}_r(k^2 - k^3). \quad . \quad . \quad . \quad (34c)$$

The student will recognize (34c) as the ordinary "general" form of Clapeyron's "Theorem of Three Moments" * derived in a somewhat different manner in Mechanics of Materials. Since no restriction was placed on r in the development, eqs. 34a, 34b, 34c will apply to *any* three consecutive supports in a continuous girder system, and the equations can be set up without direct reference to the general method expressed in eqs. 29. It is evident that the equation is directly applicable to a very large class of problems. The form (34b) can be readily ex-

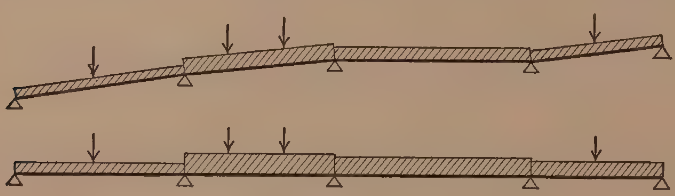


FIG. 78

tended to cover other types of loading, and recalling how the form (34a) was developed, it is evident that the case of variable moment of inertia may be provided for in a given case without difficulty. The student should note that the development does not require the supports to be originally level. The theorem applies to the system of Fig. 78a, as well as to 78b, so long as the supports fit the unstrained profile of the beam and there is complete continuity in the construction.

The theorem of three moments can be derived without recourse to the principle of work or to the general method of analysis for indeterminate stresses presented here. As a matter of fact, it was discovered and widely used before the development of this latter method. But the discussion of the preceding paragraphs should aid in making clear the setting and significance of the three moment equations in the general theory.

55. Rigid Frame with Columns Fixed.—A solution of this problem in one form was presented in Chapter II, page 112. We shall show here that

* Comptes Rendus (1857). Some authorities attribute priority of discovery to Bertot (1853), but the principle has always borne Clapeyron's name.

a fairly simple artifice in the arrangement of the statically determined base system leads to *independent* equations for the three redundants.

We shall imagine the structure divided symmetrically and the redundant forces X applied to the ends of a rigid arm as shown in Fig. 79*b*.

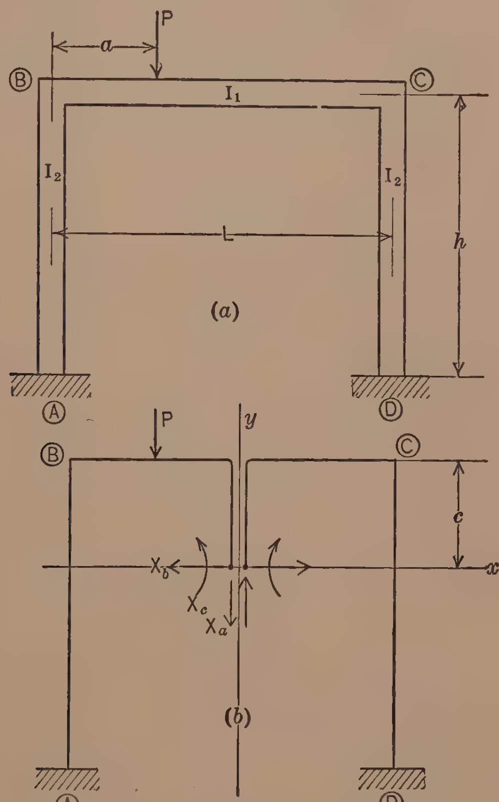


FIG. 79

The true moment, shear and thrust at the center of the horizontal member BC will then be

$$M = X_c - c \cdot X_b,\ V = X_a,\ H = X_b.$$

The general equations may be written

X_a	X_b	X_c	Constant
δ_{aa}	δ_{ab}	δ_{ac}	$= \delta'_a$
δ_{ba}	δ_{bb}	δ_{bc}	$= \delta'_b$
δ_{ca}	δ_{cb}	δ_{cc}	$= \delta'_c$

From Maxwell's principle, $\delta_{ab} = \delta_{ba}$, etc.

If M' = moment at any section of frame due to given loading, redundants removed, and

m_a = moment at any section due to X_a = unity,
m_b and m_c being similarly defined, we shall have

$$\delta'_a = \sum\int\frac{M'm_a ds}{EI} = \int_{\frac{L}{2}-a}^{\frac{L}{2}}\frac{P\left(x+a-\frac{L}{2}\right)xdx}{EI_1} + \int_0^h\frac{Pa\frac{L}{2}dy}{EI_2}$$

$$= +\frac{Pa}{2E}\left[\frac{a}{6I_1}(3L-2a)+\frac{hL}{I_2}\right];$$

$$\delta'_b = \sum\int\frac{M'm_b ds}{EI} = \int_{\frac{L}{2}-a}^{\frac{L}{2}}\frac{P\left(x+a-\frac{L}{2}\right)cdx}{EI_1} + \int_0^h\frac{Pa(c-y)dy}{EI_2}$$

$$= +\frac{Pa}{2E}\left(\frac{ac}{I_1}-\frac{(h-2c)h}{I_2}\right);$$

$$\delta'_c = \sum\int\frac{M'm_c ds}{EI} = -\int_{\frac{L}{2}-a}^{\frac{L}{2}}\frac{P\left(x+a-\frac{L}{2}\right)dx}{EI_1} - \int_0^h\frac{Pady}{EI_2}$$

$$= -\frac{Pa}{2E}\left(\frac{2h}{I_2}+\frac{a}{I_1}\right);$$

$$\delta_{aa} = \sum\int\frac{m_a^2 ds}{EI} = 2\left[\int_0^{\frac{L}{2}}\frac{x^2dx}{EI_1}+\int_0^h\frac{L^2}{4}\frac{dy}{EI_2}\right]$$

$$= \frac{L^3}{12EI_1}+\frac{hL^2}{2EI_2} = \frac{L^2}{2E}\left[\frac{L}{6I_1}+\frac{h}{I_2}\right];$$

$$\delta_{bb} = \sum\int\frac{m_b^2 ds}{EI} = 2\left[\int_0^{\frac{L}{2}}\frac{c^2dx}{EI_1}+\int_0^h\frac{(c-y)^2dy}{EI_2}\right]$$

$$= \frac{2}{E}\left[c^2\left(\frac{L}{2I_1}+\frac{h}{I_2}\right)+\frac{h^2}{I_2}\left(\frac{h}{3}-c\right)\right];$$

$$\delta_{cc} = \sum\int\frac{m_c^2 ds}{EI} = 2\left[\int_0^{\frac{L}{2}}\frac{dx}{EI_1}+\int^h\frac{dy}{EI_2}\right] = \frac{L}{EI_1}+\frac{2h}{EI_2}.$$

Now, from the symmetry of the unit loadings $X_b = 1$, $X_c = 1$ and the anti-symmetry of the unit loading $X_a = 1$ it is clear from inspection

(see Fig. 80) that $\delta_{ac} = \delta_{ca} = \delta_{ab} = \delta_{ba} = 0$. (The student may also easily verify this fact by the general formula.) We have further

$$\delta_{bc} = \delta_{cb} = \sum \int \frac{m_c m_b ds}{EI} = \frac{2}{E}\left[\int_0^{\frac{L}{2}} \frac{c dx}{I_1} + \int_0^h \frac{(c-y)dy}{I_2}\right]$$
$$= \frac{1}{E}\left[\frac{Lc}{I_1} + \frac{2ch - h^2}{I_2}\right].$$

The length of the rigid arm "c" through which we have supposed the loads applied to the substitute structure of Fig. 79b is entirely arbitrary; by a proper variation in the forces X we can maintain the

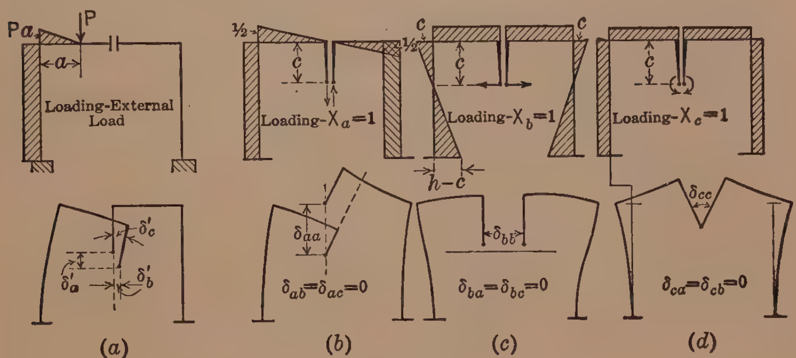


FIG. 80

desired condition with any length of arm. We propose to choose a length which will render $\delta_{bc} = o$. Letting $I_1 = kI_2$

$$\delta_{bc} = \frac{1}{EI_2}\left(\frac{Lc}{k} + 2ch - h^2\right) = 0,$$

whence

$$c = \frac{h^2}{2h + \dfrac{L}{k}}.$$

When this value of c is used all δ's vanish except δ_{aa}, δ_{bb} and δ_{cc} and we have

3aL −

$$X_a = -\frac{\delta'_a}{\delta_{aa}} = \frac{Pa}{L}\frac{3aL - 2a^2 + 6hLk}{L^2 + 6hLk};$$

$$X_b = -\frac{\delta'_b}{\delta_{bb}} = \frac{Pa}{L}\frac{3}{2}\cdot\frac{(L-a)}{h^2k + 2hL};$$

$$X_c = -\frac{\delta'_c}{\delta_{cc}} = \frac{Pa}{L}\frac{aL + 2hLk}{2(L + 2hk)}.$$

Moment diagrams and graphs showing distortions for the various cases are shown in Fig. 80*a* . . . *d*.

This method is applicable to any type of frame with fixed supports, including the case of the fixed arch (see Fig. 81*b*). In the unsymmetrical case (Fig. 81*c*) the length and direction of the auxiliary arm and the direction of one of the resolved forces, say X_b, are to be determined by the conditions that (1) the forces X_b and X_a will cause no angular change at O (from which it must follow that X_c will cause no linear displacement at O), and (2) the direction of, say, force X_b must be so determined that X_a will cause no displacement along its line of action (whence X_b will cause none in the direction X_a). The location of the point O may always be determined readily enough, since O lies at the center of gravity of the elastic weights, $\frac{L}{EI}$ or $\frac{ds}{EI}$, if the individual members have variable moments of inertia. This point is sometimes called the "elastic center" of the framework. To satisfy condition (2) we must first determine the direction of the displacement of O due to $X_a = 1$. The direction of X_b will obviously lie normal to this displacement. The method will be further illustrated in Chapter on Arches.

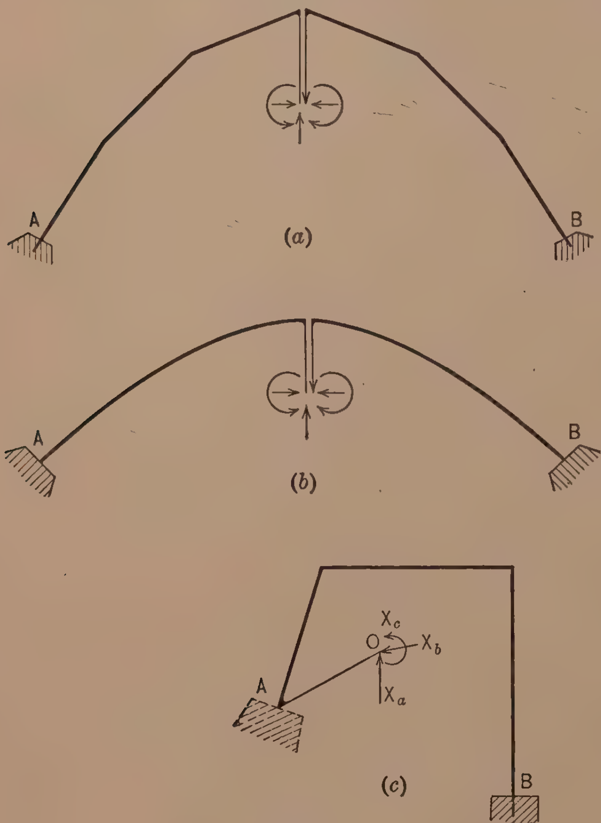


FIG. 81

56. Statically Undetermined Base-System.—It will sometimes be advantageous to work with a statically *undetermined* base-system for

which a complete and simple solution is ready to hand. The framework of Fig. 82 is five-fold statically indeterminate, hence in the ordinary course of solution five equations each containing five unknowns would be involved. We may, however, use the framework of 82*b*, i.e., a rigid

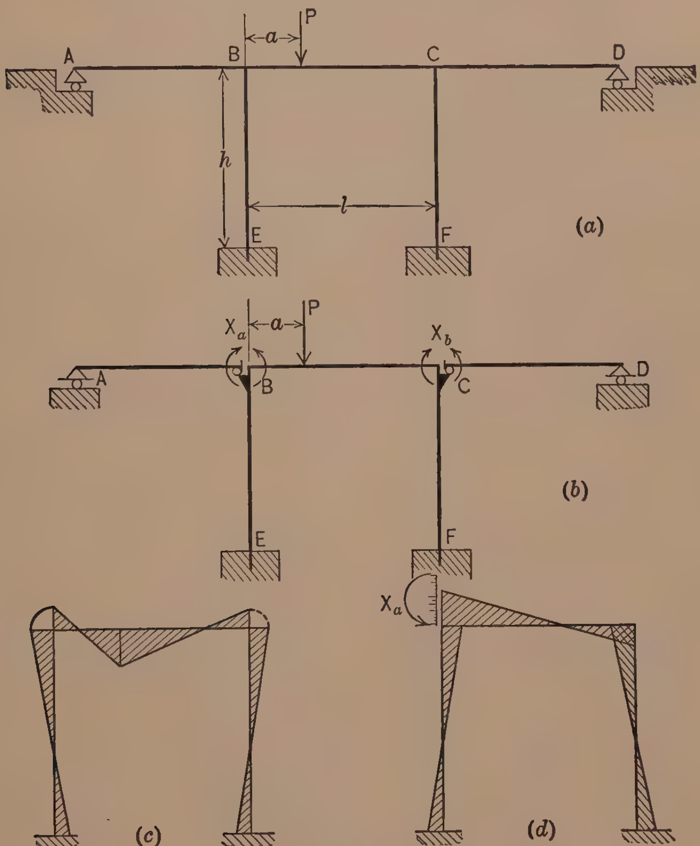


FIG. 82

rectangular frame and two simple beams, as a substitute structure. The equations will then be

$$X_a\delta_{aa} + X_b\delta_{ab} = -\ \delta'_a$$

$$X_a\delta_{ba} + X_b\delta_{bb} = -\ \delta'_b.$$

Where δ'_a = the angular change between the tangent at B of simple beam AB and the tangent at B of the cross girder, BC, *of the rigid frame* $EBCF$, due to load P, and δ_{aa}, δ_{ab}, etc., are correspondingly defined. This of course gives a vastly simpler solution *provided* we can readily

determine the angular rotation of the joints B and C in the frame $EBCF$ due to a load P, and to an applied moment acting at B or C. We recall that since E and F are fixed, the angular rotation at B and C must equal the area of the moment diagram for EB and FC. The δ''s and δ's in the above equations are then very easily obtained so soon as we know the moments at base and top of columns. The preceding example has shown that complete general formulas are readily obtainable for the statically unknown quantities in any fixed rectangular frame. Further, such formulas may be found ready to hand in a number of reference works.* This method of solution will therefore prove of great advantage in certain frame problems.

In other problems it may be advantageous to use other types of statically undetermined base systems—the two-hinged arch or the rectangular frame with hinged bases. Speaking generally the method will have unique advantage when, and only when, the statically indeterminate substitute structure possesses a reasonably simple general solution, either known in advance or readily available from tables.

SECTION III.—THE DIRECT APPLICATION OF THE MOMENT AREA PRINCIPLE

57. General Relationships.—We have emphasized in the earlier chapters that the solution of a statically indeterminate structure may always be viewed as a problem in consistent distortions. The common method of applying the law of consistent distortions is to resolve the structure into a base system (usually determinate) to which, in addition to the given loading, the redundants are applied as external forces in such a manner as to secure the required consistency of distortions. This method has been illustrated in the immediately preceding pages and in Chapter II. But it is not always necessary to formally resolve the structure into a fundamental system and redundants in order to apply the law of consistent deflections. We may note the frame of Fig. 83 for example. This is five-fold statically indeterminate. If we observe the sketch (Fig. 83*b*) showing qualitatively the distortion of the structure, it is at once evident (since there must be a common tangent at ② and since joints ①, ③ and ④ are fully fixed) that

$$\frac{\Delta_{1-2}}{L_1} = \frac{\Delta_{3-2}}{L_3}, \quad \ldots \ldots \quad (a)$$

$$= \frac{\Delta_{4-2}}{h}, \quad \ldots \ldots \quad (b)$$

$$\Delta_{2-1} = 0, \quad \ldots \ldots \quad (c)$$

$$\Delta_{2-3} = 0, \quad \ldots \ldots \quad (d)$$

$$\Delta_{2-4} = 0. \quad \ldots \ldots \quad (e)$$

* See for example, Wilson, Richart & Weiss, "Analysis of Statically Indeterminate Structures by Slope-Deflection Method," Bulletin 108, University of Illinois Experiment Station, pages 60–64; or A. Kleinlogel, "Rahmen Formeln," pages 96 and 227.

These relationships may be evaluated in terms of the moments at once by the principle of moment areas.

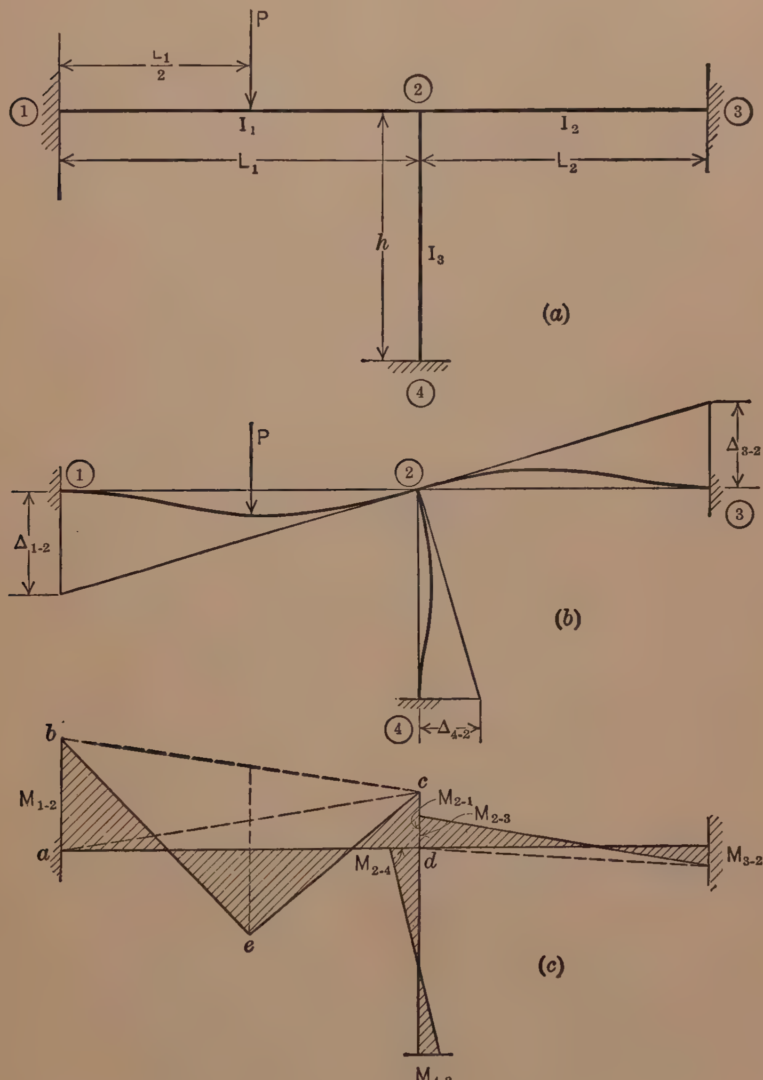


FIG. 83

58. Solution of Rigid Frame.—We have, noting that the moment of the $\frac{M}{EI}$ diagram *abecd* is equal to the moment of the triangle *bec* minus

the moment of the trapezoid *abcd*, which latter may be resolved into the $\triangle$ *bca* and *acd*,

$$\frac{PL_1^3}{16I_1} - \frac{M_{1-2}L_1}{6I_1} - \frac{M_{2-1}L_1}{3I_1} = \frac{M_{3-2}L_2}{6I_2} - \frac{M_{2-3}L_2}{3I_2} \quad . \quad . \quad (a)$$

$$= \frac{M_{4-2}h}{6I_3} - \frac{M_{2-4}h}{3I_3}, \quad . \quad . \quad . \quad (b)$$

$$\frac{PL_1^2}{16I_1} - \frac{M_{1-2}L_1}{3I_1} - \frac{M_{2-1}L_1}{6I_1} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

$$-\frac{M_{2-3}L_2}{2I_2}\cdot\frac{L_2}{3} + \frac{M_{3-2}L_2}{2I_2}\cdot\frac{2L_2}{3} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

$$-\frac{M_{2-4}h}{2I_3}\cdot\frac{h}{3} + \frac{M_{4-2}h}{2I_3}\cdot\frac{2h}{3} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

whence

$$M_{2-3} = 2M_{3-2},$$

and

$$M_{2-4} = 2M_{4-2}.$$

We have also the statical equation—

$$\Sigma M_{\text{about } ②} = 0 = M_{2-1} + M_{2-3} + M_{2-4}.$$

From these six equations we may readily solve for the six end moments.

If $K_1 = \frac{I_1}{L_1}$, $K_2 = \frac{I_2}{L_2}$ and $K_3 = \frac{I_3}{h}$, and if we denote any moment as positive which tends to rotate the corresponding joint *clockwise*

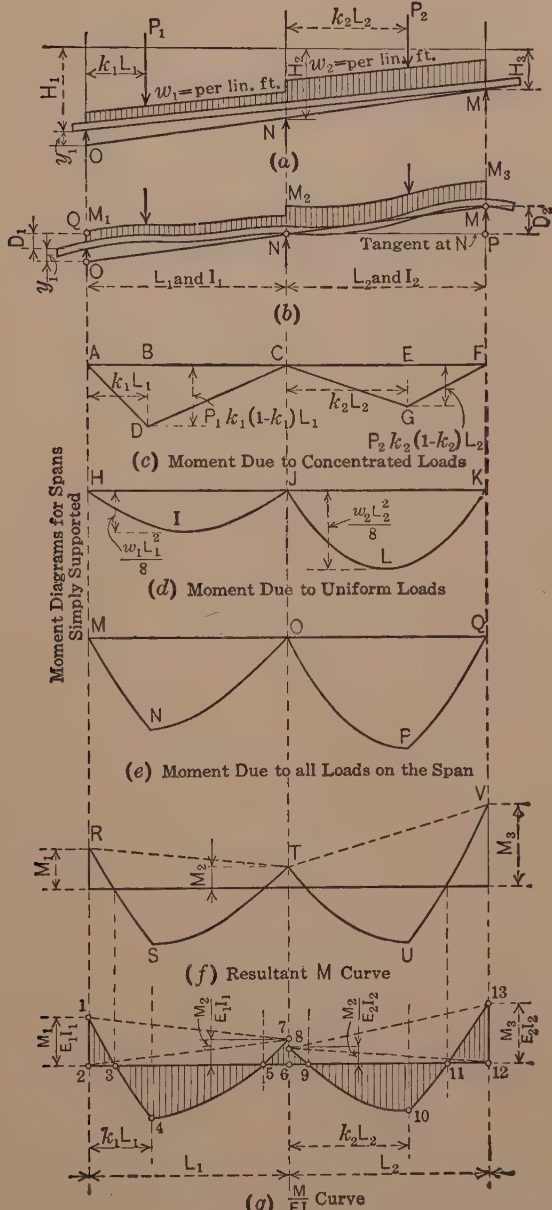
$$M_{2-4} = \frac{PL_1}{8}\,\frac{K_3}{K_1 + K_2 + K_3}, \qquad M_{2-3} = \frac{PL_1}{8}\,\frac{K_2}{K_1 + K_2 + K_3},$$

$$M_{2-1} = \frac{PL_1}{8}\,\frac{K_2 + K_3}{K_1 + K_2 + K_3}, \qquad M_{4-2} = \frac{PL_1}{16}\,\frac{K_3}{K_1 + K_2 + K_3},$$

$$M_{3-2} = \frac{PL_1}{16}\,\frac{K_2}{K_1 + K_2 + K_3}, \qquad M_{1-2} = \frac{PL_1}{16}\left[2 + \frac{K_1}{K_1 + K_2 + K_3}\right].$$

We thus see that problem is completely solved very simply and expeditiously by expressing by means of the moment area method the relations arising from the geometry of distortion.

59. Alternative Derivation of General Three Moment Theorem.— We may further illustrate the direct application of the moment area principle by the derivation of the general three moment equation. We shall take the case illustrated in Fig. 84, where the supports are "out of level," i.e., the unstrained beam does not rest on the three supports, and where E and I are different in adjacent spans.



(a)
(b)
Tangent at N
L_1 and I_1
L_2 and I_2
Moment Diagrams for Spans Simply Supported
(c) Moment Due to Concentrated Loads
(d) Moment Due to Uniform Loads
(e) Moment Due to all Loads on the Span
(f) Resultant M Curve
(g) $\frac{M}{EI}$ Curve

FIG. 84

With the notation of Fig. 84a we get from similar triangles

$$\frac{H_1 + y_1 - H_2}{L_1} = \frac{H_2 - H_3}{L_2},$$

whence

$$y_1 = \frac{H_2 - H_3}{L_2}L_1 - (H_1 - H_2). \quad . \quad . \quad . \quad . \quad (a)$$

From Fig. 84b, if QNP is the common tangent at N, after the beam is subjected to bending,

$$\frac{OQ}{QN} = \frac{MP}{PN}, \quad \text{or} \quad \frac{D_1 + y_1}{L_1} = -\frac{D_2}{L_2},$$

whence

$$y_1 = \frac{L_1}{L_2}D_2 + D_1 = \frac{H_2 - H_3}{L_2} \cdot L_1 - (H_1 - H_2)\text{—from } (a), \quad . \quad . \quad (b)$$

$$\therefore \; L_1D_2 + L_2D_1 = (H_3 - H_2)L_1 + (H_1 - H_2)L_2. \quad . \quad . \quad . \quad . \quad (c)$$

The values of D_1 and D_2, respectively the deflection of the support points O and M from a tangent to the elastic curve of ONM at N, may be readily evaluated by the principle of moment areas. We have

D_1 = Statical moment of $\dfrac{M}{EI}$ area 1–2–3–4–5–6–7 of Fig. 84g about point 2

= Moment of trapezoid 1–2–6–7 minus moment of 1–4–7

= Moment of ⚠ 1–7–2 and 7–2–6 minus $\dfrac{1}{EI_1}$ (mom. of ADC + mom. HIJ, Figs. c and d).

The values of these area moments are

Moment of area 1–7–2

$$= \frac{M_1}{E_1I_1}\left(\frac{L_1}{2}\right)\frac{L_1}{3} = \frac{M_1L_1^2}{6E_1I_1},$$

Moment of area 7–2–6

$$= \frac{M_2}{EI_1}\left(\frac{L_1}{2}\right)\left(\frac{2}{3}L_1\right) = \frac{M_2L_1^2}{3E_1I_1},$$

Moment of area ADC

$$= BCD + ABD$$

$$= \frac{P_1k_1(1 - k_1)L_1}{2E_1I_1}\left[(1 - k_1)L_1\left(k_1 + \frac{1 - k_1}{3}\right)L_1 + k_1L_1\frac{2k_1L_1}{3}\right]$$

$$= \frac{P_1L_1^3}{6E_1I_1}(k_1 - k_1^3).$$

Moment of area

$$HIJ = \frac{w_1L_1^2}{8E_1I_1}\cdot\frac{2}{3}L_1\cdot\frac{L_1}{2} = \frac{w_1L_1^4}{24E_1I_1}$$

$$\therefore\ D_1 = \frac{L_1^2}{6E_1I_1}\left[M_1 + 2M_2 - P_1L_1(k_1 - k_1^3) - \frac{w_1L_1^2}{4}\right].$$

In an entirely similar manner we get

$$D_2 = \frac{L_2^2}{6E_2I_2}\left[M_3 + 2M_2 - P_2L_2(2k_2 - 3k_2^2 + k_2^3) - \frac{w_2L_2^2}{4}\right].$$

Substituting in eq. (c) we get

$$\frac{L_1L_2^2}{6E_2I_2}\left[M_3 + 2M_2 - P_2L_2(2k_2 - 3k^2 + k_2^3) - \frac{w_2L_2^2}{4}\right]$$
$$+ \frac{L_2L_1^2}{6E_1I_1}\left[M_1 + 2M_2 - P_1L_1(k_1 - k_1^3) - \frac{w_1L_1^2}{4}\right]$$
$$= (H_3 - H_2)L_1 + (H_1 - H_2)L_2. \quad \ldots \quad (e)$$

This is the general form of Clapeyron's three moment equation. In most practical cases $E_1 = E_2$; assuming this and giving the negative sign to the moments over the supports, we have

$$- M_1\frac{L_1}{I_1} - 2M_2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) - M_3\frac{L_2}{I_2}$$
$$= \frac{w_1L_1^3}{4I_1} + \frac{w_2L_2^3}{4I_2} + \frac{P_1L_1^2}{I_1}(k_1 - k_1^3) + \frac{P_2L_2^2}{I_2}(2k_2 - 3k_2^2 + k_2^3)$$
$$+ 6E\left[\frac{H_3 - H_2}{L_2} + \frac{H_1 - H_2}{L_1}\right]. \quad \ldots \quad (f)$$

This is the ordinary form of the generalized three moment theorem.

SECTION IV.—THE SLOPE-DEFLECTION METHOD

60. General Statement.—In dealing with isolated beam problems we commonly speak of the ends either as "freely supported" or as "fixed." But the case of an intermediate condition, a partial fixity often arises, even in isolated beams, while in rigid frames it is the common case. If we take any beam in which there is a degree of restraint at the ends (Fig. 85) it is clear that the flexure of the beam is fully determined so soon as the end moments, M_A and M_B are known. These moments will depend (1) upon the given loads, (2) upon the rotation of the end tangents, and (3) upon the relative displacement of the supports. In studying the flexure of a beam from this point of view it is advantageous to take the case of fixed-ends as the standard basic condition. For

this case the end moments are determined by well-known general formulas. If there is tangential rotation or relative change of level of supports, the moments will be modified accordingly.

61. Development of Slope-Deflection Equations.—The analytical expression for the relation between the end moments and the end displacements may be derived as follows (see Fig. 86):

Imagine the simple beam AB acted upon by moments, M_A and M_B inducing angular rotations of the end tangents α_A and α_B. Further

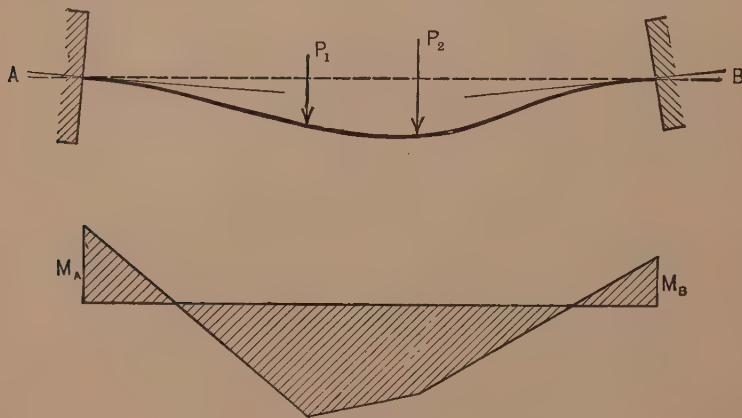


Fig. 85

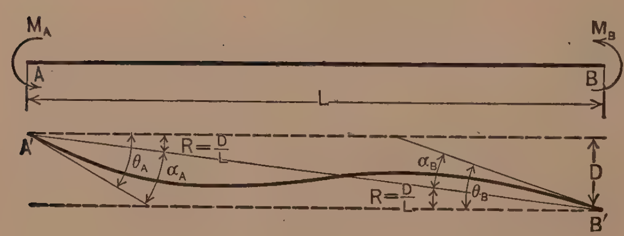


Fig. 86

assume that supports A and B are displaced to A' and B', thus inducing an angular shift in the axis of the beam $= \dfrac{D}{L} = R$. Then the total tangential change $= \theta = \alpha + R$. The angular change α may be evaluated in terms of the moments either by the method of work as in Chapter I, or by the method of moment areas * or of elastic weights.

* The slope-deflection equation is conveniently proved by the moment area principle, but, notwithstanding statements to the contrary that have appeared in the literature of the subject, there is no more necessary relation between the two than between the moment area principle and the three moment equation.

Recalling that the angular change at the end of a simple beam is numerically equal to the corresponding *end shear* in the beam when loaded with the $\frac{M}{EI}$ diagram, we get at once (considering clockwise rotation positive, and the end moments positive when acting as shown)

$$\alpha_A = \frac{-L}{6EI}(2M_A - M_B); \qquad \alpha_B = \frac{-L}{6EI}(2M_B - M_A),$$

whence

$$\theta_A = \frac{-L}{6EI}(2M_A - M_B) + R; \quad \theta_B = \frac{-L}{6EI}(2M_B - M_A) + R.$$

Solving for M_A and M_B we get

$$\left.\begin{aligned} M_A &= \frac{2EI}{L}(-2\theta_A - \theta_B + 3R) \\ M_B &= \frac{2EI}{L}(-2\theta_B - \theta_A + 3R) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (35a)$$

We have here the end moments expressed as functions of the end distortions. That is to say, if the ends of a beam are forcibly displaced by amounts θ and R, these "applied" end distortions will awaken resisting moments as indicated by eqs. 35*a* (see Fig. 87). If when these end displacements occur the beam is also acted upon by any set of loads, the principle of super-position justifies the direct combination of the different effects, i.e., the end moment will equal the ordinary fixed beam moment increased or decreased by the moment due to the end displacements. Eqs. 35*a* then become

$$\left.\begin{aligned} M_A &= M_{FA} + \frac{2EI}{L}(-2\theta_A - \theta_B + 3R) \\ M_B &= M_{FB} + \frac{2EI}{L}(-2\theta_B - \theta_A + 3R) \end{aligned}\right\} \quad . \quad . \quad . \quad (35)$$

These relations are perfectly general and apply equally to an isolated beam and to any member of a framework acting as a beam. Since they state the final value of the end moments in any given case in terms of the known fixed beam moments and the changes in slope and the relative deflections at the points of support, Eqs. (35) are commonly known as the "*slope-deflection*" equations.

Where it is desired to compute the effect on a beam of any sort of settlement of foundations, the Eqs. (35*a*) apply directly; the observed or estimated numerical values of the displacements of the supports

are substituted for the θ's and R in the right-hand member of the equation and the resulting moment follows at once.

But by far the most important application of the slope-deflection method is in the analysis of stresses in multiply statically indeterminate

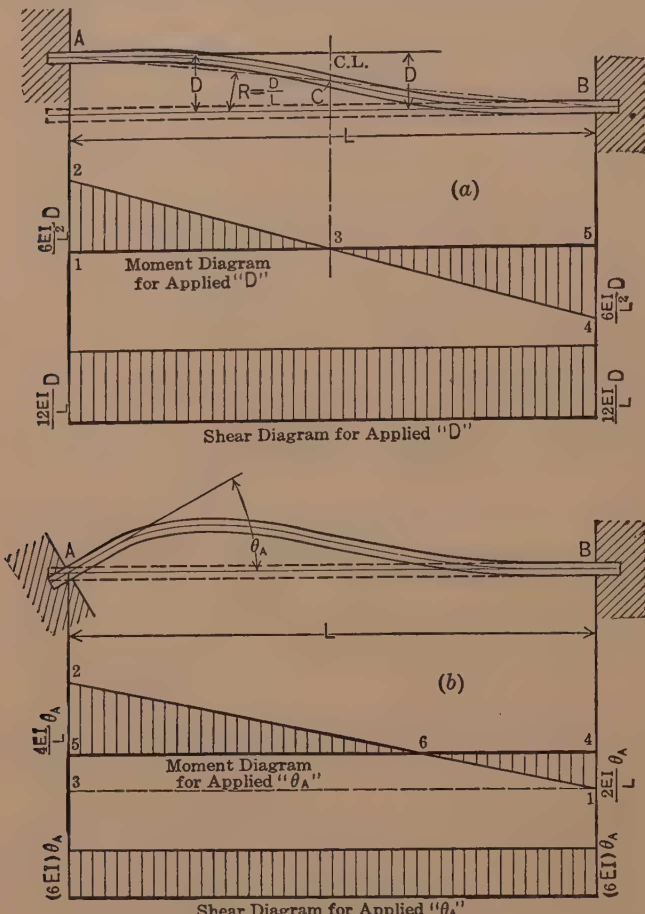


FIG. 87

structures under any given load conditions, *where the slopes and deflections are taken as the unknowns for which a solution is sought.* This use of the slope-deflection equations can best be explained through a few simple examples.

62. Application to Continuous Girder with Fixed Ends.—(Fig. 88.) Consider the two-span continuous girder ABC with ends A and C fixed

and unyielding supports. The structure is triply statically indeterminate. From the conditions just stated we have at once that

$$\theta_A = \theta_c = R = 0,$$

and the equations for the four end moments are given below the figure.

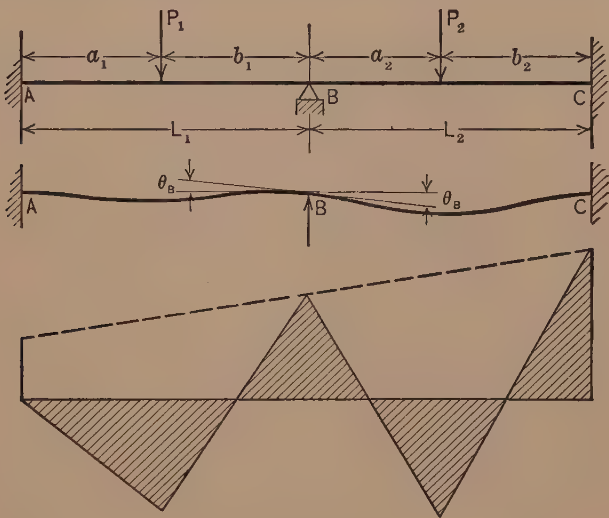


FIG. 88

$$M_{AB} = M_F + \frac{2EI_1}{L_1}[-2\theta_A - \theta_B + 3R] = \frac{+P_1a_1{}^2b_1}{L_1{}^2} + 2EK_1[-\theta_B]. \quad . \quad . \quad (1)$$

$$M_{BA} = M_F + \frac{2EI_1}{L_1}[-2\theta_B - \theta_A + 3R] = \frac{-P_1a_1b_1{}^2}{L_1{}^2} + 2EK_1[-2\theta_B]. \quad . \quad (2)$$

$$M_{BC} = M_F + \frac{2EI_2}{L_2}[-2\theta_B - \theta_C + 3R] = \frac{+P_2a_2{}^2b_2}{L_2{}^2} + 2EK_2[-2\theta_B]. \quad . \quad (3)$$

$$M_{CB} = M_F + \frac{2EI_2}{L_2}[-2\theta_C - \theta_B + 3R] = \frac{-P_2a_2b_2{}^2}{L_2{}^2} + 2EK_2[-\theta_B]. \quad . \quad . \quad (4)$$

$$M_{BA} + M_{BC} = 0; \quad \therefore \theta_B = \frac{-1}{4E(K_1 + K_2)}\left[\frac{P_1a_1b_1{}^2}{L_1{}^2} - \frac{P_2a_2{}^2b_2}{L_2}\right]$$

$$\therefore\ M_{AB} = +\frac{Pa_1{}^2b_1}{L_1{}^2} - \frac{1}{2}\frac{K_1}{K_1 + K_2}\left[\frac{P_1a_1b_1{}^2}{L_1{}^2} - \frac{P_2a_2{}^2b_2}{L_2{}^2}\right]$$

$$M_{BC} = +\frac{Pa_2{}^2b_2}{L_2{}^2} - \frac{K_2}{K_1 + K_2}\left[\frac{P_1a_1b_1{}^2}{L_1{}^2} - \frac{P_1a_2{}^2b_2}{L_2{}^2}\right], \quad \text{etc.}$$

Equilibrium about joint B requires that $M_{BA} = M_{BC}$, whence the value of θ_B is easily determined, and this value substituted in eqs. ① to ④ gives

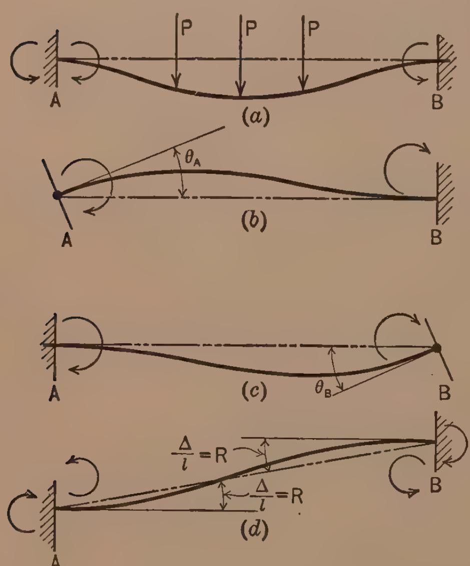
the value for each end moment. Thus the solution of three simultaneous equations for the statically undetermined moments which would be required by the general method of Chapter II, and also by the three-moment theorem and by the direct application of the moment area method, is entirely avoided. Two points should be noted: (1) the slight modification in notation and (2) the significance of the sign convention.

(1) Where several members enter the same joint, say M, it is necessary in order to avoid ambiguity to specify the end moments in the different members by a double subscript, thus,

$$M_{FMN} = M_{MN} + \frac{2EI_{MN}}{L_{MN}}[-2\theta_M - \theta_N + 3R_{MN}],$$

$$M_{FNM} = M_{NM} + \frac{2EI_{NM}}{L_{NM}}[-2\theta_N - \theta_M + 3R_{NM}],$$

where M_{MN} = moment at M in beam MN, etc.



$\theta_A = \theta_B = R = 0$; Loads applied to beam; $M_A = +$; $M_B = -$.

$P = 0$; $\theta_B = R = 0$; Negative θ_A induced in beam; $M_A = +$; $M_B = +$.

$P = 0$; $\theta_A = R = 0$; Negative value of θ_B induced in beam; $M_A = +$; $M_B = +$.

$P = 0$; $\theta_A = \theta_B = 0$; Negative value of R induced in beam; $M_A = -$; $M_B = -$.

FIG. 89

For most cases we may omit the subscripts for M_F and for R without loss of clearness.

(2) The sign convention (see Fig. 89) is most important in the application of the slope-deflection method. θ and $R\left(= \frac{D}{L}\right)$ are angles

measured in radians; they are to be taken as positive when the angular movement is clockwise. The end moments M are taken as positive when they tend to rotate the *joint* on which they act (*not* the *member*) * in a clockwise direction. Fig. 90 shows the beams of the preceding problem cut away and the moments acting on the joints (the end support is also treated as a "joint"). The signs are indicated according to the rule just stated. It should be noted carefully that the ordinary sign convention in which a moment is treated as positive or negative according to whether the stress in the top fiber is compression or tension has no application here.

FIG. 90

Since beginners in the use of the slope-deflection method frequently find especial difficulty "keeping the signs straight," the student is advised to study carefully the simple rules stated above, applying them to easy examples until thorough mastery is obtained. Their application is invariable, and once they are mastered all difficulty with signs in slope-deflection analysis disappears.

63. Application to Rectangular Frame (*a*).—(See Fig. 91.) If we assume that axial deformation in the three-legged bent with fixed bases may be neglected, we have $R_{CD} = 0$, $R_{CA} = R_{DB}$, and with load applied as shown, $M_F = 0$ for all members. The remainder of the solution is indicated in full on the figure. PH is taken as positive when it corresponds to a positive R. Since we take the moments M_{AC}, M_{CD}, etc., as the moments *acting on the joint*, the minus sign must be used in the equation (1a) which sums up the moments acting on the members AC and BD. Since the shear across the bent $= P$, we must have the summation of the four end moments equal to the moments of the end shears (Fig. 91*c*).

64. Application to Rectangular Frame (b).—It will be interesting to compare the solution by the slope-deflection method of the frame of Fig. 83, pp. 142–4, with that previously worked out by the direct application of the moment-area principle. Noting that

$$\theta_1 = \theta_3 = \theta_4 = R = 0,$$

* A different sign convention might equally well be used—see for example, Wilson, Richart and Weiss, "Solution of Statically Indeterminate Structures by the Slope-Deflection Method"—University of Illinois, Engineering Experiment Station Bulletin No. 108. The convention adopted here has appeared to the authors to have some advantage in simplicity.

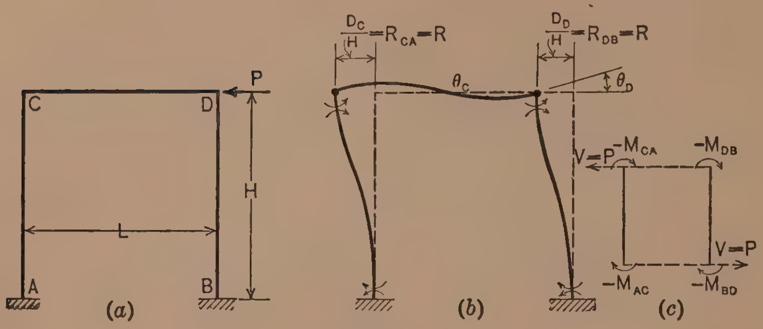


FIG. 91

$$\theta_A = \theta_B = 0;\ D_C = D_D. \quad \therefore\ R_{CA} = R_{DB} = R.$$

If $\dfrac{I}{L} = K$, we have:

$$M_{AC} = -2EK_{A\text{-}C}[\theta_C - 3R];\quad M_{BD} = -2EK_{B\text{-}D}[\theta_D - 3R].$$

$$M_{CA} = -2EK_{A\text{-}C}[2\theta_C - 3R];\quad M_{DB} = -2EK_{B\text{-}D}[2\theta_D - 3R]$$

$$M_{CD} = -2EK_{C\text{-}D}[2\theta_C + \theta_D];\quad M_{DC} = -2EK_{CD}[2\theta_D + \theta_C]$$

$$-M_{AC} - M_{CA} - M_{BD} - M_{DB} + PH = 0; \qquad (1a)$$

$$\therefore\ -2EK_{AC}[3\theta_C + 3\theta_D - 12R] = PH, \text{ (if } K_{AC} = K_{BD}),$$

and

$$R = \frac{\theta_C + \theta_D}{4} + \frac{PH}{24\,EK_{AC}}. \qquad (1)$$

Also,

$$M_{CA} + M_{CD} = 0, \quad \text{whence} \quad R = \frac{2(K_{AC} + K_{CD})\theta_C + K_{CD}\theta_D}{3K_{AC}}. \qquad (2)$$

Likewise,

$$M_{DB} + M_{DC} = 0, \quad \text{whence} \quad R = \frac{2(K_{BD} + K_{CD})\theta_D + K_{CD}\theta_C}{K_{BD}}. \qquad (3)$$

From (2) and (3) we have $\quad \theta_C = \theta_D \qquad (4)$

whence,

$$R = \frac{2K_{AC} + 3K_{CD}\theta_C}{3K_{AC}} = (C + \tfrac{2}{3})\theta_C\left(\frac{K_{CD}}{K_{AC}} = C\right). \qquad (5)$$

$\therefore$ Substituting (5) in (1) and recalling (4) we have

$$\theta_c = \frac{PH}{4EK_{AC}}\cdot\frac{1}{1 + 6c}. \qquad (6)$$

Substituting values from (5) and (6) we get,

$$M_{CA} = -M_{CD} = -M_{DC} = M_{DB} = \frac{PH}{2}\left(\frac{3C}{6C + 1}\right);$$

$$M_{AC} = \frac{PH}{2}\cdot\frac{1 + 3C}{1 + 6C}.$$

the equations for the six moments become (denoting θ_2 simply as θ and $\frac{I}{L}$ as K)

$$M_{1-2} = +\frac{PL_1}{8} - 2EK_1\theta \qquad M_{3-2} = -2EK_2\theta$$

$$M_{2-1} = -\frac{PL_1}{8} - 4EK_1\theta \qquad M_{2-3} = -4EK_2\theta$$

$$M_{2-4} = -4EK_3\theta \qquad M_{4-2} = -2EK_3\theta.$$

Since

$$M_{2-1} + M_{2-3} + M_{2-4} = 0,$$

we have

$$\theta = -\frac{PL_1}{32E}\frac{1}{K_1 + K_2 - K_3},$$

whence by direct substitution,

$$M_{1-2} = \frac{PL_1}{16}\left(2 + \frac{K_1}{K_1 + K_2 + K_3}\right),$$

$$M_{3-2} = \frac{PL_1}{16}\left(\frac{K_2}{K_1 + K_2 + K_3}\right),$$

$$M_{2-1} = -\frac{PL_1}{8}\left(\frac{K_2 + K_3}{K_1 + K_2 + K_3}\right),$$

$$M_{2-3} = \frac{PL_1}{8}\left(\frac{K_2}{K_1 + K_2 + K_3}\right),$$

$$M_{2-4} = \frac{PL_1}{8}\left(\frac{K_3}{K_1 + K_2 + K_3}\right),$$

$$M_{4-2} = \frac{PL_1}{16}\left(\frac{K_3}{K_1 + K_2 + K_3}\right).$$

The signs given are in accord with the sign convention stated on page 153, i.e., the moment at a joint is positive if the moment applied alone would tend to rotate the joint clockwise.

65. Summary.—Of all the special methods for simplifying the solution of certain types of indeterminate structures which we have discussed in this chapter, the slope-deflection method has by far the widest practical application and will be used more frequently in the following chapters than any other. The unique feature of the method is that it breaks up the structure, however complex, into a set of relatively simple partially restrained beams, stating the end moments in terms of the known fixed beam moments and the effects due to slope changes and deflections, and then derives a set of equations *using these slopes and deflections as the unknowns* instead of the statically undetermined end moments.

The advantage of the method lies in the fact that in very many cases the unknown slopes and deflections are markedly fewer than the unknown moments and therefore the number of equations to be solved simultaneously is proportionately reduced. In example 1 we had a triply indeterminate structure but only *one* unknown slope; in example 3 we had but one unknown slope for a five-fold indeterminate frame. The greater the number of fixed supports and the number of members entering a single joint, the greater the advantage of the slope-deflection method over other methods of solution. It is applicable to any rigid-joint framework (so far as bending of the members is concerned), and for most such problems, with the exception of continuous girders, where the three-moment theorem is the favored method, it is believed that it will furnish the simplest solution. For analyzing frames of the type of Fig. 83 and Fig. 92, the slope-deflection method is incomparably

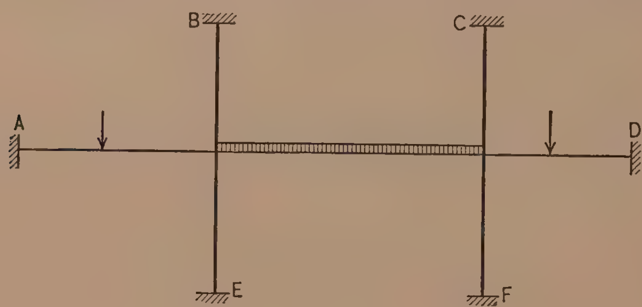


FIG. 92

superior to any other. For a case such as the frame of Fig. 91, where no reduction in the number of unknowns to be solved for results, the ease and directness, indeed, the nearly mechanical character of the application of the equations is noteworthy. It must not be supposed that the method will always show to such marked advantage. In the frame of Fig. 91, if the column bases are hinged, the structure is singly indeterminate, and the general method gives a rapid and easy solution. On the contrary the slopes and deflections are increased from three to five and thus the method is at a clear disadvantage.

Many examples of the application of the slope-deflection method to the numerical solution of actual structures will be found in Chapter V.

Historical Note. The use of distortions instead of moments, stresses or reactions as the unknowns to work for in a statically indeterminate analysis is at least as old as Manderla's original solution of the secondary stress problem proposed in 1878. Manderla, however, used the angle between the end tangent and the axis of the strained member as the

unknown rather than the angle which the tangent itself turns through (see H. Manderla . . . Allgemeine Bauzeitung, 1880). The use of the slope-deflection equation dates from Mohr's publication of his solution for secondary stresses in 1892–3 (see O. Mohr, Zivilinginieur, Dec. 1892–Jan. 1893). This application is limited to the case where the member carries no load between the end points. The first presentation of the slope-deflection method (as here defined) as a general method of attack on rigid joint problems is due to George A. Maney. (See Engineering Studies #1, University of Minnesota, 1915.)

Recently Professor Ostenfeld of Copenhagen has presented a still more general aspect of the method under the title of the "Method of Deformations," applying to all cases where the anlaysis sets out to obtain distortions rather than moments, stresses or reactions. (See A. Ostenfeld, "Die Deformationsmethode," Der Bauingenieur, (Berlin) Jan. 31, 1923.)

CHAPTER IV

CONTINUOUS GIRDERS

66. Preliminary.—In the broadest sense the continuous girder includes all girders, solid or framed, which rest on more than two supports. A single beam with fixed ends may be regarded as a three-span continuous beam with the end spans indefinitely shortened.

The types commonly met with in American practice are:

(A) Restrained Beams. An isolated girder with fully fixed ends is not a common type of structure, but partially restrained beams both as independent girders and as members of a composite framework are encountered very frequently. In many cases the end conditions cannot be determined with certainty, and the fixed-end moments are desired as a limiting case. Further, we have seen that the method of analysis by slope-deflections uses the fully restrained beam as the basic condition of every member. On account of these facts, the theory of the restrained beam is perhaps the most important section, *practically*, of the continuous-girder theory.

(B) Floor Systems in Building Construction. In both steel frame and reinforced concrete buildings continuity of construction is the general practice, and an accurate analysis of such structures requires them to be treated as rigid frames. When, however, the restraining effect of the columns is slight, or when only roughly approximate results are desired, the floor girders may be treated as multi-span continuous beams.

(C) Continuous Steel Bridges:

1. Turntables.
2. Swing Bridges:
   - (a) Two-span center bearing bridge (solid girder or truss).
   - (b) Three-span rim bearing bridge (solid girder or truss—usually latter and partially continuous over center span).
3. Long span continuous trusses.

(*D*) In some cases the continuous girder theory may be applied to advantage to a portion of a structure not primarily designed as such. Thus the bending stresses in the riveted top chord of a bridge truss, arising from the elastic deflection of the truss under loads (secondary stresses), may be obtained approximately by treating the chord as a continuous girder under no loads, but in which a displacement of each joint is forcibly imposed on the girder.

The present chapter will be devoted to a consideration of:

I. The fully restrained beam under various types of loading, with several numerical examples.

II. The general treatment of the multi-span continuous girder, illustrating the application of the three-moment theorem in its various forms, the construction of influence lines and numerical examples.

III. Continuous and swing bridges, including two- and three-span swing bridges, the partially continuous girder, and influence lines. A complete numerical example of the stress analysis for a swing bridge is appended.

SECTION I.—THE FULLY RESTRAINED BEAM

67. Equation for End Moments.—*Concentrated Load.*—The end moments in a fixed beam are easily deduced from the moment area principle (see Fig. 93). Since the deflection of A from a tangent at B and the deflection of B from a tangent at A each equal zero, we have

$$\frac{L^2}{6EI}(2M_B + M_A) = \frac{PL^3}{6EI}(2k - 3k^2 + k^3),$$

and

$$\frac{L^2}{6EI}(2M_A + M_B) = \frac{PL^3}{6EI}(k - k^3),$$

whence

$$M_A = PL(k^2 - k^3). \quad . \quad . \quad . \quad . \quad . \quad (36a)$$

In Fig. 93 kL is measured from B; evidently if $k' = 1 - k$ we shall have

$$M_B = PL(k'^2 - k'^3), \quad . \quad . \quad . \quad . \quad . \quad (36b)$$

where k' is measured from A. If $P = 1$, equation (36a) is the influence line for M_A (see Fig. 94a).

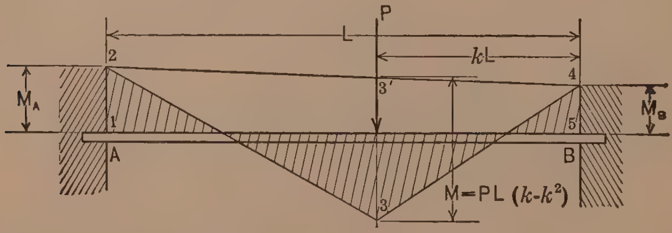
68. *Uniform Load.*—If we wish to get the effect of a broken load w uniformly distributed from $x = k_2L$ to $x = k_1L$ (see Fig. 94b), we may

sum up the load per unit times the corresponding area under the influence line thus

$$M_A = \sum_{k_2L}^{k_1L} w\Delta(kL)\cdot L(k^2 - k^3) = wL^2\int_{k_2}^{k_1}(k^2 - k^3)dk$$

$$= wL^2\left[\frac{k^3}{3} - \frac{k^4}{4}\right]_{k_2}^{k_1}. \quad . \quad . \quad (37)$$

If $k_2 = 0$, that is if the load w extends from B to a distance kL, equation (37) may be viewed as the influence line for M_A where a load w per linear foot is substituted for a concentrated load of unity. Fig. 94*b* shows this influence line. To get the value of M_A for a broken load (from k_2L to k_1L) we simply take the difference between corresponding values of $\left(\frac{k^3}{3} - \frac{k^4}{4}\right)$ and multiply by wL^2.



The area moment of 2–3–4 about (A) equals:

$$\frac{PL}{EI}(k - k^2)\left[\frac{1-k}{2}L \times \tfrac{2}{3}(1 - k)L + \frac{kL}{2} \times (1 - \tfrac{2}{3}k)L\right],$$ taking 2–3–3′ and 4–3–3′ separately.

$$\therefore \frac{PL^3}{2EI}(k - k^2)[\tfrac{2}{3}(1 - k)^2 + k(1 - \tfrac{2}{3}k)] = \frac{PL^3}{6EI}(k - k^2)(2 - k)$$

$$= \frac{PL^3}{6EI}(2k - 3k^2 + k^3).$$

Likewise the area moment of 2–3–4 about (B) equals:

$$\frac{PL^3}{2EI}(k - k^2)\left[(1 - k)\left(k + \frac{(1-k)}{3}\right) + k\left(\frac{2k}{3}\right)\right]$$

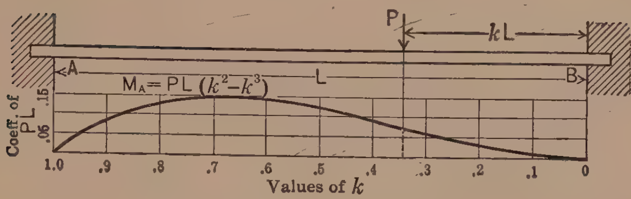
$$= \frac{PL^3}{6EI}(k - k^2)(1 + k) = \frac{PL^3}{6EI}(k - k^3).$$

The moment area of 1–2–4–5 about (A) equals:

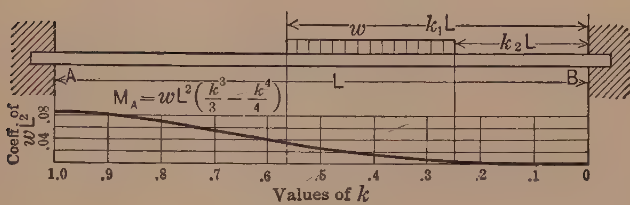
$$\frac{L^2}{2EI}[\tfrac{2}{3}M_B + \tfrac{1}{3}M_A] \quad \text{or} \quad \frac{L^2}{6EI}(2M_B + M_A).$$

Likewise the moment area of 1–2–4–5 about (B) equals: $\frac{L^2}{6EI}(2M_A + M_B)$.

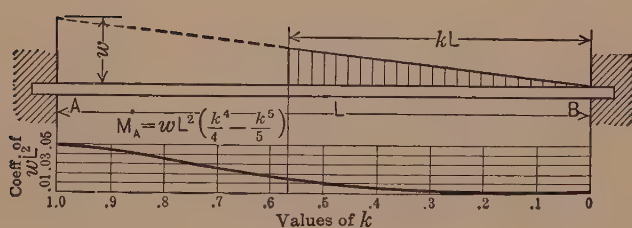
Fig. 93.



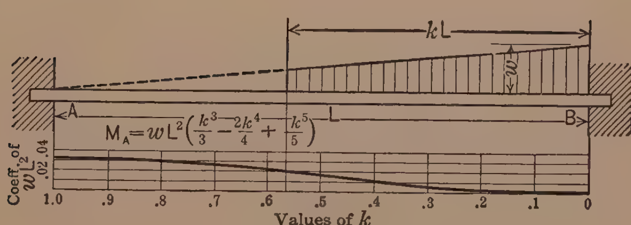
(*a*) Curve for $k^2 - k^3$



(*b*) Curve for $\dfrac{k^3}{3} - \dfrac{k^4}{4}$



(*c*) Curve for $\dfrac{k^4}{4} - \dfrac{k^5}{5}$



(*d*) Curve for $\dfrac{k^3}{3} - \dfrac{2k^4}{4} + \dfrac{k^5}{5}$

Fig. 94

TABLE II

Decimal	Fra.	Coefficients of PL: $k^2 - k^3$	Coefficients of wL^2: $\frac{k^3}{3} - \frac{k^4}{4}$	Coefficients of wL^2: $\frac{k^4}{4} - \frac{k^5}{5}$	Coefficients of wL^2: $\frac{k^3}{3} - \frac{2k^4}{4} + \frac{k^5}{5}$
.000	...	.0000	.00000	.00000	.00000
.050	...	.0024	.00004	.00000	.00004
.100	...	.0090	.00031	.00002	.00030
.111	$\frac{1}{9}$	.0109	.00042	.00003	.00038
.125	$\frac{1}{8}$	.0137	.00059	.00005	.00054
.143	$\frac{1}{7}$	.0175	.00087	.00009	.00078
.167	$\frac{1}{6}$	.0232	.00136	.00017	.00120
.200	$\frac{1}{5}$	.0320	.00227	.00034	.00193
.222	$\frac{2}{9}$	.0384	.00304	.00050	.00254
.250	$\frac{1}{4}$	.0469	.00423	.00078	.00350
.286	$\frac{2}{7}$	.0584	.00612	.00129	.00483
.300	...	.0630	.00698	.00154	.00544
.333	$\frac{1}{3}$	.0740	.00923	.00226	.00697
.350	...	.0796	.01054	.00270	.00784
.375	$\frac{3}{8}$	.0879	.01264	.00346	.00918
.400	$\frac{2}{5}$	.0960	.01493	.00435	.01058
.428	$\frac{3}{7}$	.1048	.01774	.00551	.01223
.444	$\frac{4}{9}$	.1096	.01946	.00625	.01320
.450	...	.1114	.02012	.00656	.01356
.500	$\frac{1}{2}$	.1250	.02604	.00938	.01667
.556	$\frac{5}{9}$	.1372	.03343	.01324	.02019
.572	$\frac{4}{7}$	.1401	.03562	.01452	.02110
.600	$\frac{3}{5}$	.1440	.03960	.01684	.02276
.625	$\frac{5}{8}$	.1465	.04324	.01904	.02420
.650	...	.1479	.04691	.02140	.02551
.667	$\frac{2}{3}$	.1482	.04943	.02308	.02635
.700	...	.1470	.05431	.02640	.02791
.715	$\frac{5}{7}$	.1457	.05651	.02797	.02854
.750	$\frac{3}{4}$	.1406	.06152	.03170	.02982
.778	$\frac{7}{9}$	.1344	.06536	.03460	.03076
.800	$\frac{4}{5}$	.1280	.06827	.03686	.03141
.833	$\frac{5}{6}$	.1159	.07230	.04017	.03213
.850	...	.1084	.07420	.04170	.03250
.857	$\frac{6}{7}$	.1050	.07497	.04243	.03254
.875	$\frac{7}{8}$	.0957	.07677	.04403	.03274
.889	$\frac{8}{9}$	.0878	.07805	.04514	.03291
.900	...	.0810	.07898	.04582	.03316
.950	...	.0451	.08217	.04883	.03334
1.000	...	.0000	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{30}$

By a similar procedure we find that for a uniformly increasing (triangular) distributed load having a value of 0 at B and w at A, the equation for M_A is

$$M_A = wL^2\left(\frac{k^4}{4} - \frac{k^5}{5}\right). \quad \quad (38)$$

(See Fig. 94c.)

This is equivalent to taking $P = k$ in equation (36*a*) and integrating from zero to k. Obviously if we have a broken load which extends from k_2L to k_1L varying directly with k, we shall get

$$M_A = wL^2 \left[\frac{k^4}{4} - \frac{k^5}{5}\right]_{k_2}^{k_1}. \quad \quad (38a)$$

Likewise for a loading of the type shown in Fig. 94*d*

$$M_A = wL^2\left(\frac{k^3}{3} - \frac{2k^4}{4} + \frac{k^5}{5}\right), \quad \quad (38b)$$

and if such a loading extends from k_2L to k_1L

$$M_A = wL^2 \left[\frac{k^3}{3} - \frac{2k^4}{4} + \frac{k^5}{5}\right]_{k_2}^{k_1}. \quad \quad (38c)$$

The functions of k in these four cases, shown graphically in Figs. 94*a* to 94*d*, are given numerically in Table II. They are of great aid in solving numerical problems, particularly special and more or less irregular cases, as the following examples will illustrate.

69. Example 1.—(See Fig. 95.) We have here the simultaneous application of three different types of loading, (*a*) unequal concentrations

EXAMPLE 1

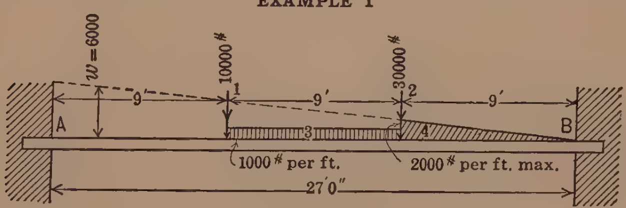


FIG. 95.

TABLE A

Due to Load	Value of K		Value of C		Moment in Foot-Pounds		Shear in Pounds	
	A	*B*	*A*	*B*	*A*	*B*	*A*	*B*
1	$\frac{2}{3}$	$\frac{1}{3}$	.148	.074	40,000	20,000	6,670	3,330
2	$\frac{1}{3}$	$\frac{2}{3}$	.074	.148	60,000	120,000	10,000	20,000
3	$\frac{1}{3}$ to $\frac{2}{3}$	$\frac{1}{3}$ to $\frac{2}{3}$	.0402	.0402	29,300	29,300	4,500	4,500
4	$\frac{1}{3}$	$\frac{2}{3}$ to 1	.00226	.0070	9,900	30,600	2,000	7,000
Due to diff. in end moments.......					....	....	−2,250	+2,250
Totals......................					139,200	199,900	20,920	37,080

at the $\frac{1}{3}$ point, (*b*) uniform load over middle third and (*c*) a uniformly varying load over the outer third on one side.

Entering Table II with proper values of k for the ends A and B (note that for A, k is measured from B; for B it is measured from A) for each loading, we get directly (or by a simple subtraction) the values C of the coefficients which multiplied by PL or wL^2 give the moments M_A and M_B. Table A gives all the results.

For the shear calculation we note that the final shear may be obtained by combining the simple beam shear with that due to the end moments M_A and M_B. If the moments are taken positive when causing compression on upper fiber and the end shear positive when acting upward on the portion of beam outside of the section, we must have that the shear due to combined end moments is

$$V_A{}^M = \frac{M_B - M_A}{L}; \quad V_B{}^M = \frac{M_A - M_B}{L}.$$

The simple beam shears are

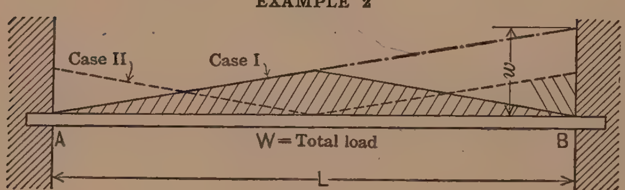
$$V'_A = 23170; \quad V'_B = 34830.$$

The shears due to end moments are

$$V_A{}^M = \frac{-199900 - (-139200)}{27} = -2250,$$

$$V_B{}^M = \frac{-139200 - (-199900)}{27} = +2250.$$

EXAMPLE 2



CASE I

M_A due to load on right half $= .00938\ wL^2$.
M_A due to load on left half $= .01667\ wL^2$.
M_A due to total load W $= .02605\ wL^2$.

$$\text{But } W = \frac{wL}{4} \text{ or } w = \frac{4W}{L}$$

$$\therefore M = .1042\ WL.$$

CASE II

$$M = .0624\ WL$$

Fig. 96

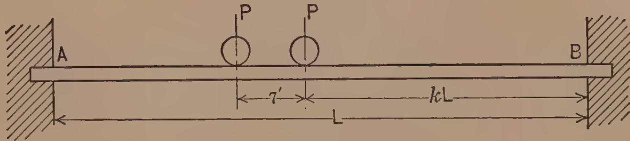
The final shears therefore are

$$V_A = 23170 - 2250 = 20920,$$

$$V_B = 34830 + 2250 = 37080.$$

70a. Example 2.—(Fig. 96.) This treats of two cases of special symmetrical loading, worked out very simply by means of Table II.

EXAMPLE 3



(a)

$$M_A = PL\left[k^2 - k^3 + \left(k + \frac{7}{L}\right)^2 - \left(k + \frac{7}{L}\right)^3\right] \quad . \quad . \quad . \quad . \quad (1)$$

$\dfrac{dM_A}{dk}$ is the tangent to the moment influence line as the loads P, P move across the span from B to A. When tangent passes through zero, $\dfrac{dM_A}{dk} = 0$. This gives the value of k for the position of the loads to produce a maximum moment at A.

Determination of K for Maximum Moment at A

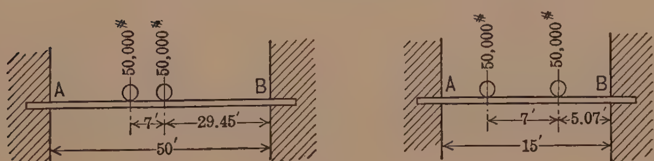
$$\frac{dM_A}{dk} = 0 = PL\left[2k - 3k^2 + 2\left(k + \frac{7}{L}\right) - 3\left(k + \frac{7}{L}\right)^2\right]$$

$$-3k^2 - 3k^2 + 2k + 2k - \frac{42}{L}k - \frac{147}{L^2} + \frac{14}{L} = 0$$

$$k^2(-6) + k\left(4 - \frac{42}{L}\right) - \frac{147}{L^2} + \frac{14}{L} = 0$$

$$k^2 - \left(\frac{4L - 42}{6L}\right)k - \left(\frac{14L - 147}{6L^2}\right) = 0$$

$$k = \frac{2L - 21}{6L} + \sqrt{\left(\frac{2L - 21}{6L}\right)^2 + \frac{14L - 147}{6L^2}} \quad . \quad . \quad . \quad . \quad (2)$$



(b) (c)

k by (2) $= .589$
M_A by (1) $= 708800$ ft.-lbs.

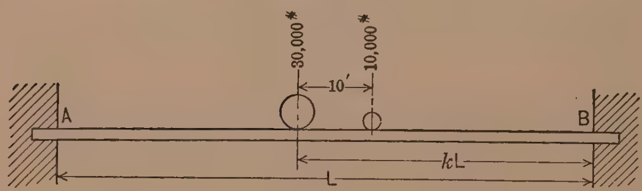
k by (2) $= .338$
M_A by (1) $= 76{,}300$ ft.-lbs.

Fig.—97.

Case I is a triangular loading with a maximum unit value at the center; $M_A = M_B = .1042\,WL$ if W = total load. Case II is the same total load distributed oppositely, i.e., w varies symmetrically from zero at center to a maximum at the ends. Evidently the sum of the loadings I and II is a uniformly distributed total load of $2W$, for which the end moment is $\frac{1}{6}WL$. For Case II then

$$M_A = M_B = \frac{WL}{6} - .1042WL = .0624WL.$$

EXAMPLE 4



(a)

$$M_A = 30{,}000L(k^2 - k^3) + 10{,}000L\left[\left(\frac{kL-10}{L}\right)^2 - \left(\frac{kL-10}{L}\right)^3\right]. \quad . \quad (1)$$

The maximum value of M_A is to be determined as in Ex. 3.

DETERMINATION OF k FOR MAXIMUM MOMENT AT A

$$\frac{dM_A}{dk} = 0 = 3(2k - 3k^2) + \left[2\left(k - \frac{10}{L}\right) - 3\left(k - \frac{10}{L}\right)^2\right]$$

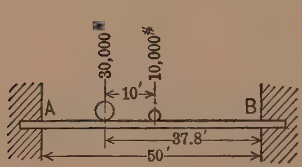
$$6k - 9k^2 + 2k - \frac{20}{L} - 3k^2 + \frac{60k}{L} - \frac{300}{L^2} = 0$$

$$6kL^2 - 9k^2L^2 + 2kL^2 - 20L - 3k^2L^2 + 60kL - 300 = 0$$

$$12L^2k^2 - (8L^2 + 60L)k + 20L + 300 = 0$$

$$K^2 - \left(\frac{8L^2 + 60L}{12L^2}\right)k + \left(\frac{20L + 300}{12L^2}\right) = 0$$

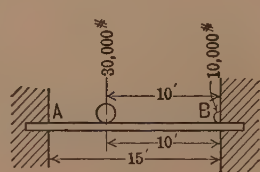
$$k = \frac{2L + 15}{6L} + \sqrt{\left(\frac{2L+15}{6L}\right)^2 - \left(\frac{5L + 75}{3L^2}\right)}. \quad . \quad . \quad . \quad . \quad (2)$$



(b)

k by (2) = .757

M_A by (1) = 275,300 ft.-lbs.



(c)

k by (2) = .6667

M_A by (1) = 66,600 ft.-lbs.

FIG. 98

70b. Example 3.—(Fig. 97.) We have here a conventional railway bridge loading, two heavy moving loads P on axles 7′ 0″ apart. Equation ② (on the figure) gives the criterion for the position causing a maximum end moment. Two numerical cases are appended; $P =$ 50,000# in each case, and $L = 50'$ in one and 15′ in the other.

70c. Example 4.—(Fig. 98.) This is similar to Example 3, except that the loading is the conventional 20-ton tractor highway bridge loading. The criterion for maximum moment (for the loading shown) is developed on the figure, and the numerical results for the same two spans as in Example 3 are shown. Both these problems are very quickly solved by Table II as soon as the critical value of k is determined.

70d. Example 5.—(Fig. 99.) The loading indicated in this problem is a conventional railway bridge loading sometimes used as an alternate to Cooper's E-40. To locate the position for a maximum, moments for three trial locations were plotted as ordinates against k as abscissae, and the maximum determined as shown in Fig. 99(b). This method is readily applied to *any* type of loading whatsoever, and should be used where it is inconvenient to derive an algebraic criterion as was done in Examples 3 and 4.

SECTION II.—THEORY OF MULTI-SPAN CONTINUOUS GIRDER

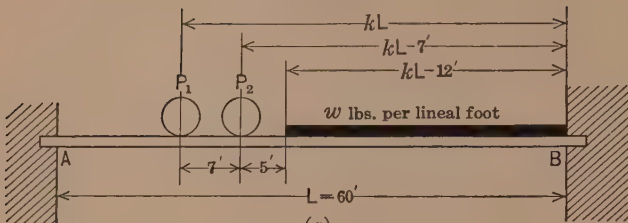
71. General Considerations.—It is obvious that the analysis of any continuous girder is reduced to a simple beam problem so soon as the moments at the support are known. The true moment diagram for any system of loads will be the ordinary simple beam moment diagram combined with the moment diagram due to a set of external moments equal to the support moments acting as applied loads on the series of spans treated as simple beams. Fig. 100 shows the two sets of moment diagrams and their combination. The shears are obtained from the formula

$$V_1 = V'_1 + \frac{M_2 - M_1}{L_2}$$

The problem is thus completely solved when the support moments are determined.

Either the three-moment theorem or the slope-deflection equations will serve as a general method by which any continuous-girder problem may be solved. If the end supports are fully fixed slope-deflections may be applied to advantage, but otherwise the three-moment method is the most expeditious. We will illustrate the application of this method by the following problems.

EXAMPLE 5



(a)

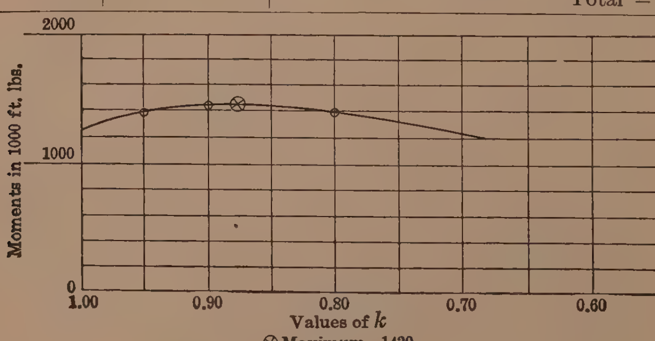
w = 4000 lbs. per lineal foot.
$P_1 = P_2$ = 50,000 lbs.
$P_{1-2}L = 50 \times 60 = 3000$ (Moment in 1000 ft.-lbs.).
$wL^2 = 4 \times 3600 = 14{,}400$ (Moment in 1000 ft.-lbs.).

To determine maximum moment at A due to conventional train loading coming on from end B.

Method: Try first load (P_1) at distances (kL) from the end B equal to $.8L$, $9L$, $.95L$, respectively and plot a smooth curve through the moments and determine maximum location of P_1 by trial.

Example:—L = 60 ft. loading as shown.

	Load	Moment in 1000 Ft.-Lbs. Based on Constant from Table II
1. $kL=8\text{–}10L$	P_1	$0.1280 \times P_1L = 384$
	P_2	$0.1475 \times P_2L = 443$
	w	$0.0396 \times wL^2 = 570$
		Total = 1397
2. $kL=9\text{–}10L$	P_1	$0.0810 \times P_1L = 243$
	P_2	$0.1330 \times P_2L = 399$
	w	$.0.0543 \times wL^2 = 782$
		Total = 1424
3. $kL=95\text{–}100L$	P_1	$0.0451 \times P_1L = 135$
	P_2	$0.1159 \times P_2L = 348$
	w	$0.06152 \times wL^2 = 886$
		Total = 1369



(b)

FIG. 99.

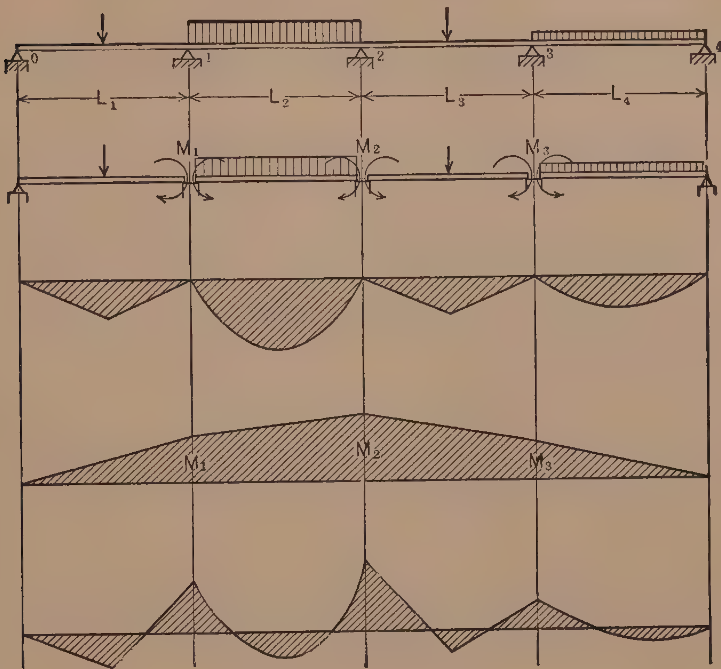


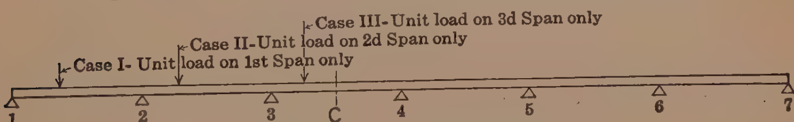
FIG. 100

72. Examples.

Problem I*a* shows the solution by means of the three moment equation for a six-span continuous girder with a single concentrated load in any span. The solution is carried through separately for the loading in each span and from these data the influence line for the moment at any section of the girder may be drawn. Such influence lines for a support point and an intermediate point are shown in Fig. 100*a*.

PROBLEM I*a*

Moment Influence Lines for Girder of 6 Equal Spans



For equal spans the typical 3-monent equation becomes —

$$M_{n-1} + 4M_n + M_{n+1} - \Sigma P_n L(k_n - k_n^3) - \Sigma P_{n+1} L(2k_{n+1} - 3k^2_{n+1} + k^3_{n+1}).$$

If we call $(k - k^3) = C_1$ and $(2k - 3k^2 + k^3) = C_2$ the 5 simultaneous equations for the moments M_2 to M_6 inclusive, may be tabulated as shown in Table A, where the resulting values are to be interpreted as coefficients of PL for the loading cases indicated. Influence lines for M_4 and M_c are shown in Fig. 100*a*. The complete detail of the solution is indicated in the following tables. C_1 and C_2 are evaluated by means of Table III.

TABLE A

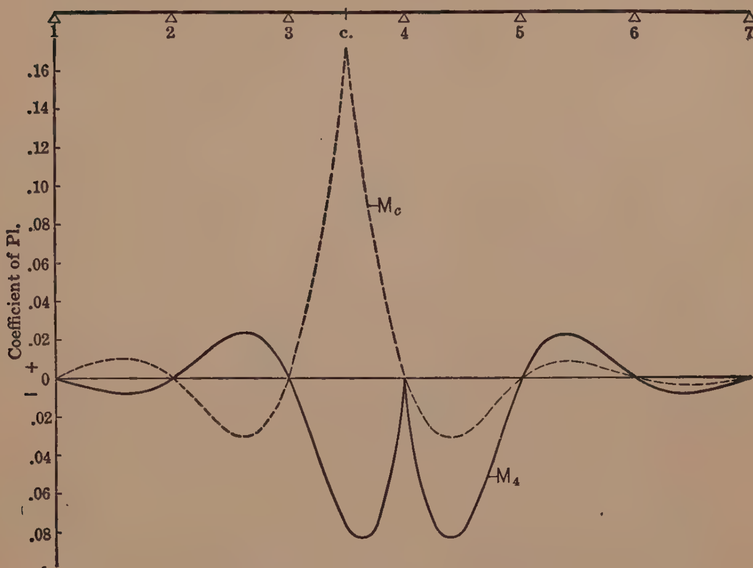
Equation	M_2	M_3	M_4	M_5	M_6	Case I	Case II	Case III
1	4	1	....	....	....	$-C_1$	$-C_2$	0
2	1	4	1	....	....	0	$-C_1$	$-C_2$
3	....	1	4	1	....	0	0	$-C_1$
4	....	....	1	4	1	0	0	0
5	....	....	....	1	4	0	0	0
a	....	....	4	16	4	0	0	0
b	....	....	4	15	....	0	0	0
c	....	15	60	15	....	0	0	$-15C_1$
d	....	15	56	....	....	0	0	$-15C_1$
e	56	224	56	....	....	0	$-56C_1$	$-56C_2$
f	56	209	....	....	....	0	$-56C_1$	$+15C_1-56C_2$
g	56	14	....	....	....	$-14C_1$	$-14C_2$	0
h	....	195	....	....	....	$+14C_1$	$+14C_2-56C_1$	$+15C_1-56C_2$
j	....	1	....	....	....	$+.072C_1$	$+.072C_2-.287C_1$	$+.077C_1-.287C_2$
k	1	....	....	....	....	$-.268C_2$	$-.268C_2+.072C_1$	$-.019C_1+.072C_2$
l	....	....	1	....	....	$-.019C_1$	$-.019C_2+.077C_1$	$-.288C_1+.077C_2$
m	....	....	....	1	....	$+.005C_1$	$+.005C_2-.021C_1$	$+.077C_1-.021C_2$
n	....	....	....	....	1	$-.001C_1$	$-.001C_2+.005C_1$	$-.019C_1+.005C_2$

TABLE B—Values of M_4

k	Case I	Case II	Case III
.1	$-.00190$	$+.00433$	$-.0149$
.2	$-.00369$	$+.00923$	$-.0325$
.3	$-.00525$	$+.01413$	$-.0503$
.4	$-.00646$	$+.01846$	$-.0663$
.5	$-.00721$	$+.02163$	$-.0783$
.6	$-.00738$	$+.02308$	$-.0831$
.7	$-.00687$	$+.02221$	$-.0812$
.8	$-.00554$	$+.01846$	$-.0678$
.9	$-.00329$	$+.01122$	$-.0414$

TABLE C—Values of M_c

k	Case I		Case II		Case III	
	Left	Right	Left	Right	Left	Right
.1	$+.0026$	$-.0007$	$-.0059$	$+.0016$	$+.0216$	$-.0055$
.2	$+.0050$	$-.0014$	$-.0126$	$+.0033$	$+.0494$	$-.0120$
.3	$+.0072$	$-.0019$	$-.0193$	$+.0052$	$+.0836$	$-.0186$
.4	$+.0088$	$-.0024$	$-.0252$	$+.0068$	$+.1241$	$-.0245$
.5	$+.0099$	$-.0026$	$-.0296$	$+.0079$	$+.1709$	$-.0288$
.6	$+.0101$	$-.0027$	$-.0315$	$+.0085$	$+.1241$	$-.0309$
.7	$+.0094$	$-.0025$	$-.0303$	$+.0081$	$+.0835$	$-.0299$
.8	$+.0076$	$-.0020$	$-.0252$	$+.0068$	$+.0494$	$-.0249$
.9	$+.0045$	$-.0012$	$-.0154$	$+.0041$	$+.0215$	$-.0152$



Influence Lines for M_c and M_4.

FIG. 100a

Problem I*b* (Fig. 100*b*) shows typical moment diagrams for concentrated loads at the center of each successive span.

PROBLEM I*b*

Moment Diagrams for Concentrated Loads—Girder with 6 Equal Spans. Data from Tables of Problem I*a*.

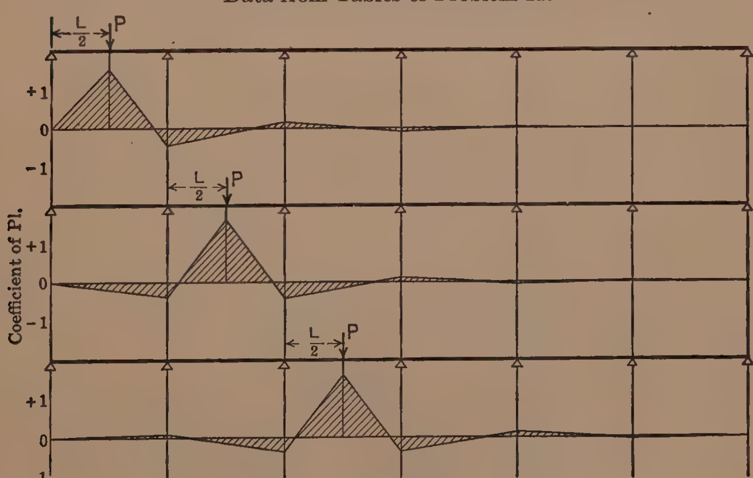


FIG. 100*b*

Problem II develops general formulae for a girder of 3 equal spans load in any manner. Tables III to VI will aid greatly in handling numerical cases.

PROBLEM II

Girder of 3 equal spans; any arrangement of concentrated or uniformly distributed loading in any span.

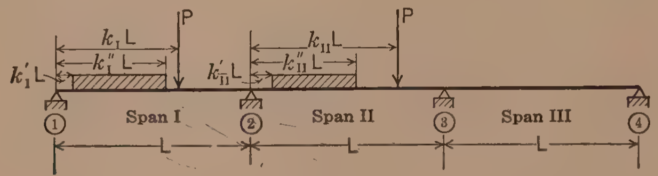


FIG. 100c

CASE I.—Span I loaded (concentrated load).
CASE II.—Span II loaded (concentrated load).

Equation	Case I	Case II
$(a) 4M_2+M_3=$	$-PL(k_I-k_I^3)$	$-PL(2k_{II}-3k_{II}^2+k_{II}^3)$
$(b) M_2+4M_3=$		$-PL(k_{II}-k_{II}^3)$
$(b') 4M_2+16M_3=$		$-4PL(k_{II}-k_{II}^3)$
$(b')-(a) 15M_3=$	$PL(k_I-k_I^3)$	$-PL(2k_{II}+3k_{II}^2-5k_{II}^3)$
$M_3=$	$\frac{1}{15}PL(k_I-k_I^3)$	$-\frac{1}{15}PL(2k_{II}+3k_{II}^2-5k_{II}^3)$
$M_2=$	$-\frac{4}{15}PL(k_I-k_I^3)$	

For uniform loads let $P = w \cdot d(kL) = wLdk$. Then, for load extending from $k'L$ to $k''L$, we have

$$M_3 = \frac{wL^2}{15}\int_{k'_I}^{k''_I}(k_I - k_I^3)dk = \frac{1}{60}wL^2(2k_I^2 - k_I^4)\Bigg|_{k'_I}^{k''_I}, \quad . \quad . \quad \text{Case I}$$

$$= \frac{-wL^2}{15}\int_{k'_{II}}^{k''_{II}}(2k_{II} + 3k_{II}^2 - 5k_{II}^3)dk = -\frac{wL^2}{60}(4k_{II}^2 + 4k_{II}^3 - 5k_{II}^4)\Bigg|_{k'_{II}}^{k''_{II}}, \text{Case II}$$

$$M_2 = -\frac{wL^2}{15}(2k_I^2 - k_I^4)\Bigg|_{k'_I}^{k''_I}$$

Tables III to VI give the values of the functions $(k - k_3)$ and $(2k - 3k^2 + k^3)$, $(2k^2 - k^4)$, $(2k + 3k^2 - 5k^3)$ and $(4k^2 + 4k^3 - 5k^4)$.

Note on use of tables:

For span III, use coefficients for span I, measuring kL from 4. For M_2 with loading in span II, use coefficients for M_3, measuring kL from 3.

Such expressions as $(2k^2 - k^4)\Big|_{k'}^{k''}$ are to be evaluated as the difference in the values of the function $(2k^2 - k^4)$ for $k = k'$ and $k = k''$.

TABLE III

Values of $k - k^3$ and $2k - 3k^2 + k^3$

$k - k^3$ (read down)

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09		
.0	.0000	.0100	.0200	.0300	.0399	.0499	.0598	.0697	.0795	.0893	.0990	.9
.1	.0990	.1087	.1183	.1278	.1373	.1466	.1559	.1651	.1742	.1831	.1920	.8
.2	.1920	.2007	.2094	.2178	.2262	.2344	.2424	.2503	.2580	.2656	.2730	.7
.3	.2730	.2802	.2872	.2941	.3007	.3071	.3134	.3193	.3251	.3307	.3360	.6
.4	.3360	.3411	.3459	.3505	.3548	.3589	.3627	.3662	.3694	.3724	.3750	.5
.5	.3750	.3773	.3794	.3811	.3825	.3836	.3844	.3848	.3849	.3846	.3840	.4
.6	.3840	.3830	.3817	.3800	.3779	.3754	.3725	.3692	.3656	.3615	.3570	.3
.7	.3570	.3521	.3468	.3410	.3348	.3281	.3210	.3135	.3054	.2970	.2880	.2
.8	.2880	.2786	.2686	.2582	.2473	.2359	.2239	.2115	.1985	.1850	.1710	.1
.9	.1710	.1564	.1413	.1256	.1094	.0926	.0753	.0573	.0388	.0197	.....	0
		.09	.08	.07	.06	.05	.04	.03	.02	.01	0	

$2k - 3k^2 + k^3$ (read up)

TABLE IV

Values of $2k^2 - k^4$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	0	.0002	.0008	.0018	.0032	.0050	.0072	.0098	.0128	.0161
.1	.0199	.0241	.0286	.0335	.0388	.0445	.0505	.0570	.0638	.0709
.2	.0784	.0863	.0945	.1030	.1119	.1211	.1306	.1405	.1506	.1611
.3	.1719	.1830	.1943	.2059	.2178	.2300	.2424	.2551	.2679	.2811
.4	.2944	.3079	.3217	.3356	.3497	.3640	.3784	.3930	.4077	.4226
.5	.4375	.4525	.4677	.4829	.4982	.5135	.5289	.5442	.5596	.5750
.6	.5904	.6057	.6210	.6363	.6514	.6665	.6815	.6963	.7110	.7255
.7	.7399	.7541	.7681	.7818	.7953	.8086	.8216	.8343	.8466	.8587
.8	.8704	.8817	.8927	.9032	.9133	.9230	.9322	.9409	.9491	.9568
.9	.9639	.9705	.9764	.9817	.9865	.9905	.9939	.9965	.9984	.9996

TABLE V

Values of $2k + 3k^2 - 5k^3$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.0000	.0203	.0412	.0672	.0843	.1070	.1298	.1532	.1767	.2008
.1	.2250	.2498	.2747	.2997	.3253	.3505	.3763	.4022	.4282	.4538
.2	.4800	.5058	.5322	.5577	.5838	.6095	.6348	.6602	.6852	.7103
.3	.7350	.7593	.7832	.8072	.8303	.8539	.8757	.8972	.9187	.9398
.4	0.9600	0.9798	0.9987	1.0172	1.0348	1.0520	1.0683	1.0837	1.0982	1.1123
.5	1.1250	1.1368	1.1482	1.1582	1.1673	1.1755	1.1828	1.1887	1.1937	1.1973
.6	1.2000	1.2013	1.2017	1.2007	1.1982	1.1945	1.1893	1.1827	1.1752	1.1658
.7	1.1550	1.1428	1.1292	1.1137	1.0968	1.0780	1.0578	1.0362	1.0122	0.9873
.8	.9600	.9313	.9002	.8677	.8333	.7970	.7583	.7182	.6757	.6313
.9	.5850	.5363	.4857	.4327	.3778	.3207	.2613	.1992	.1352	.0688

TABLE VI

Values of $4k^2 + 4k^3 - 5k^4$

	0	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	0	.0004	.0016	.0037	.0066	.0105	.0152	.0209	.0274	.0350
.1	.0435	.0530	.0635	.0750	.0875	.1010	.1155	.1311	.1477	.1653
.2	.1840	.2037	.2245	.2463	.2691	.2930	.3179	.3438	.3707	.3986
.3	.4275	.4572	.4882	.5201	.5528	.5865	.6210	.6565	.6928	.7300
.4	0.7680	0.8068	0.8464	0.8867	0.9251	0.9695	1.0119	1.0549	1.0985	1.1428
.5	1.1875	1.2327	1.2785	1.3246	1.3711	1.4180	1.4651	1.5126	1.5602	1.6080
.6	1.6560	1.7040	1.7521	1.8001	1.8481	1.8960	1.9436	1.9911	2.0383	2.0851
.7	2.1315	2.1775	2.2229	2.2678	2.3120	2.3555	2.3982	2.4401	2.4811	2.5211
.8	2.5600	2.5978	2.6345	2.6698	2.7039	2.7365	2.7676	2.7971	2.8250	2.8511
.9	2.8755	2.8979	2.9184	2.9368	2.9530	2.9670	2.9786	2.9878	2.9945	2.9986

Numerical Example:—

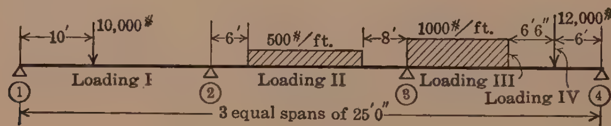


FIG. 100*d*

From preceding formulae and tables:

	M_2	M_3
Loading I	$-\frac{4}{15}PL(k_I - k_I^3)(k = .4)$ $= -\frac{4}{15}\times 10{,}000\times 25\times .3360 = -22{,}400'$ #	$+\frac{1}{15}PL(k_I - k_I^3) = +5600'$ #
Loading II	$\frac{-wL^2}{60}(4k_{II}^2 + 3k_{II}^3 - 5k_{II}^4)\Big\|_{k'=.32}^{k''=.76}$ $= -\frac{500\times 625}{60}\cdot(2.398 - .4882) = -9950'$ # NOTE:—k is measured from (3).	$-\frac{wL^2}{60}(4k_{II}^2 + 3k_{II}^3 - 5k_{II}^4)\Big\|_{k'=.24}^{k''=.68}$ $= -\frac{500\times 625}{60}\cdot(2.038 - .269)$ $= -9220'$ #
Loading III	$+\frac{wL^2}{60}(2k_{III}^2 - k_{III}^4)\Big\|_{k'=.5}^{k''=1.0}$ $= +\frac{1000\times 625}{60}\times(1 - .4375) = +5850'$ #	$-\frac{wL^2}{15}\left(2k_{III}^2 - k_{III}^4\right)\Big\|_{k'=.5}^{k''=1.0}$ $= -23{,}400'$ #
Loading IV	$+\frac{1}{15}PL(k_{III} - k_{III}^3)(k = .24) = +4520'$ #	$-\frac{4}{15}PL(k_{III} - k_{III}^3) = -18{,}000$
Total	$-22{,}000'$ #	$-45{,}000'$ #

The general procedure is identical with that shown in the preceding problems regardless of the number and lengths of spans involved; however, it is rarely necessary to attempt a complete solution of a continuous girder of more than four spans. Particular attention should be called to the rapid "dying-out" in the effect of any single load. Thus

the influence lines for M_4 and M_c in problem I*a* show that the effect of a load more than two spans removed is quite negligible.

Problem III (Fig. 100*e*) is a two-span continuous girder with fixed ends for which the slope-deflection method offers a ready solution shown in full in the figure.

PROBLEM III

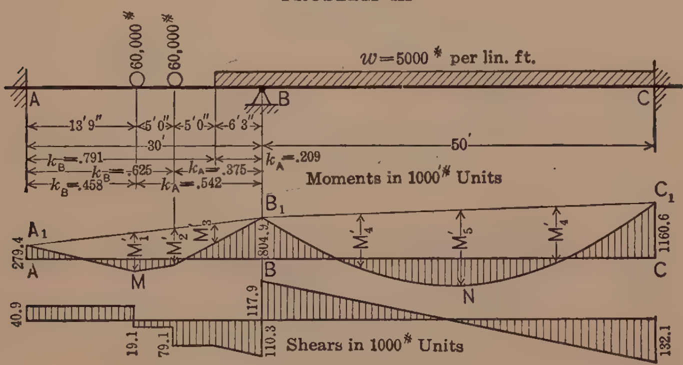


FIG. 100*e*

CALCULATION OF FIXED BEAM AND MOMENTS

$M_{FAB} = 60{,}000 \times 30(.0879 + .1340) + 5000 \times \overline{30}^2(.0026) = +411{,}200'{}^{\#}$

$M_{FBA} = 60{,}000 \times 30(.1137 + .1465) + 5000 \times \overline{30}^2(.0833 - .0670) = -541{,}300'{}^{\#}$

$M_{FBC} = M_{FCB} = \frac{1}{12}(5000)(50)^2 = \pm 1{,}042{,}000'{}^{\#}$

(See Table II for coefficients used above.)

MOMENT EQUATIONS	CALCULATION OF MOMENT VALUES $(1000'{}^{\#})$	
$M_{AB} = M_{FAB} + K(-\theta_B)$	$= +411.2 + 1.0[-1 - 31.8] = +279.4$	
$M_{BA} = M_{FBA} + K(-2\theta_B)$	$= -541.3 + 1.0[-2(131.8)] = -804.9$	Check
$M_{BC} = M_{FBC} + K_1(-2\theta_B)$	$= +1042.0 + .9[-2(131.8)] = +804.9$	
$M_{CB} = M_{FCB} + K_1(-\theta_B)$	$= -1042.0 + 9[-131.8] = -1160.6$	

K and K_1 are relative values of $\left(\frac{2EI}{L}\right)$ for AB and BC respectively.

Let $K = 1.0, \quad K_1 = 0.9$

$$M_{BA} + M_{BC} = 0 \text{ or } (K + K_1)(+2\theta_B) = M_{FBA} + M_{FBC} \therefore \theta_B = \frac{M_{FBA} + M_{FBC}}{2(K + K_1)}$$

Substituting $$\theta_B = \frac{-541.3 + 1042.0}{2(1.0 + 0.9)} = +131.38$$

Moment values from preceding solution are shown in figure as AA_1, BB_1, and CC_1. Plot moment values for simply supported case from base lines A_1B_1 and B_1C_1 in a *vertical* direction to obtain finished diagram. $M'_1 = 800$ $M'_2 = 790$ $M'_3 = 489$ $M'_4 = 1173$ $M'_5 = 1563$.

For true shear, correct shear values for simply supported beam as follows:

$$\underline{V_{AB}} = V'_{AB} + \frac{M_{BA} - M_{AB}}{L} = 58.4 + \frac{-804.9 - (-279.4)}{30} = 40.9$$

(V'_{AB} = simple beam reaction at A in AB.)

73. Special Applications.—It may be of interest to note briefly some of the simpler applications of the three-moment theorem as compared with that involved in the preceding examples.

Case (*a*).—For two equal spans, only one span loaded, I constant and supports unyielding, the general equation simplifies into

$$-M_1 - 4M_2 - M_3 = \Sigma PL(k - k^3),$$

PROBLEM—Case (*a*)

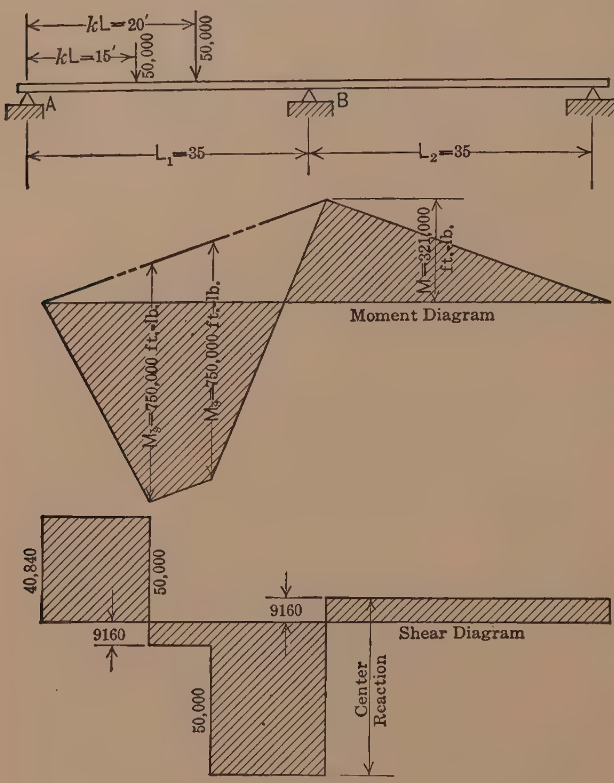


FIG. 101

$$-M_1 - 4M_2 - M_3 = P_1 1(K - K^3)$$

$$M_1 = M_3 = 0 \qquad K = \tfrac{3}{7} \text{ and } \tfrac{4}{7}.$$

(See Table III) $-4M = 50{,}000 \times 35(.349 + .385)$

$$\therefore\ M_B = -50{,}000 \times 35 \times .734 \div 4 = -321{,}000 \text{ ft.-lb.}$$

$$V_A = R_A = 50{,}000 - \frac{321{,}000}{35} = 40{,}840^{\#}$$

$$V_C = R_C = -9160^{\#}.$$

Simple beam moment $= 15 \times 50{,}000 = 750{,}000$ ft.-lb.

and since

$$M_1 = M_3 = 0,$$

$$M_2 = \Sigma\frac{PL}{4}(k - k^3).$$

Values for $(k - k^3)$ are given in Table III. Fig. 101 shows the shear and moment diagrams for numerical case.

Case (*b*).—For the case of a two-span girder under uniformly distributed load, L, I and w different in each span, we get

$$-M_1\left(\frac{L_1}{I_1}\right) - 2M_2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) - M_3\left(\frac{L_2}{I_2}\right) = \frac{w_1L_1^2}{4}\left(\frac{L_1}{I_1}\right) + \frac{w_2L_2^2}{4}\left(\frac{L_2}{I_2}\right).$$

It will be observed from the form of this equation that it is only the *relative* value of $\frac{L}{I}$ which is significant. Fig. 102 illustrates a numerical case.

PROBLEM—Case (*b*)

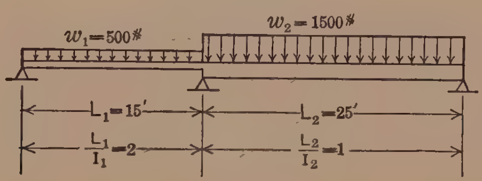


FIG. 102.

$$-M_1\left(\frac{L_1}{I_1}\right) - 2M_2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) - M_3\left(\frac{L_2}{I_2}\right) = \frac{w_1L_1^2}{4}\left(\frac{L_1}{I_1}\right) + \frac{w_2L_2^2}{4}\left(\frac{L_2}{I_2}\right)$$

$$M_1 = M_3 = 0$$

$$\therefore\ -2M_2(2+1) = \frac{500\ (15)^2}{4}\cdot\ .\ (2) + \frac{1500\ (25)^2}{4}\cdot\ .\ (1)$$

$$6M_2 = \frac{-225{,}000 - 937{,}000}{4} \qquad \therefore\ M_2 = -48{,}400 \text{ ft.-lb.}$$

Case (*c*).—Fig. 103*a* illustrates a girder of three equal spans, rigid supports and the loading and stiffness uniform throughout.

Here the general equations reduce to

$$-M_1 - 4M_2 - M_3 = \frac{wL^2}{2},$$

$$-M_2 - 4M_3 - M_4 = \frac{wL^2}{2}.$$

Since $M_1 = M_4 = 0$, and the structure is symmetrical, the value for $M_2 = M_3$ may be written at once as $-\frac{wL^2}{10}$.

Case (*d*).—Fig. 103*b* shows the same problem with a different I for the center span. The form of the equation here is

$$-\frac{M_1}{I_1} - 2M_2\frac{I_1 + I_2}{I_1I_2} - \frac{M_3}{I_3} = \frac{wL^2}{4}\frac{I_1 + I_2}{I_1I_2}.$$

The solution for any numerical case follows readily as may be seen from the figure, and we note there that increasing the stiffness of the center span to five times the others increases the moments at the ends of this span about 15 per cent.

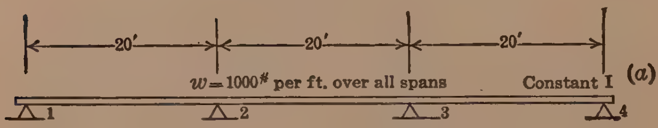
In general we have $\left(\text{if } \frac{I_1}{I_2} = k\right)$

$$M_2 = M_3 = -\frac{wL^2}{4}\cdot\frac{1 + k}{2 + 3k}.$$

If $k = 1$, we have the case of uniform stiffness, and

$$M_2 = M_3 = -\frac{wL^2}{10},$$

PROBLEM—Case (*c*)



$$-M_1 - 4M_2 - M_3 = \frac{2(1000)(20)^2}{4} \quad \ldots\ldots\ldots \quad (1)$$

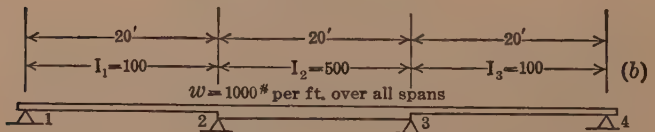
$$-M_2 - 4M_3 - M_4 = \text{do.} \quad (\text{Also } M_1 = M_4 = 0) \quad \ldots \quad (2)$$

Solving (1) and (2),

$$M_2 = M_3 = -40{,}000 \text{ ft.-lb.}$$

FIG. 103*a*

PROBLEM—Case (*d*)



$$-\frac{M_1}{100} - 2M_2\left(\frac{100 + 500}{50{,}000}\right) - \frac{M_3}{500} = \frac{1000\,(20)^2}{4}\left(\frac{100 + 500}{50{,}000}\right) \quad \ldots \quad (1)$$

and since $M_1 = M_4 = 0$

$$-M_2\left(\frac{1200}{50{,}000}\right) - \frac{M_3}{500} = 100{,}000\left(\frac{600}{50{,}000}\right)$$

From symmetry, $M_2 = M_3 = -46{,}200$ ft.-lb. as compared with $-40{,}000$ ft.-lb. for the condition of uniform stiffness.

FIG. 103*b*

as indicated in (c) above. If $k = 0$, the middle beam is infinitely stiff compared to the others, and

$$M_2 = M_3 = -\frac{wL^2}{8},$$

the end moment in a beam fully restrained at one end and free at the other.

(e) It is frequently of practical importance to estimate the effect of a slight settlement of the support on the stresses in a continuous beam. We will take the case of a 15 in. I @ 42 lbs., supported freely at the ends of a 20 ft. span, and on which we imagine a center deflection of $\frac{1}{4}$ in. to be forcibly imposed. The three moment equation applicable is

$$-M_1 - 4M_2 - M_3 = \frac{6EI}{L^2}(H_1 + H_3 - 2H_2).$$

Since

$$M_1 = M_3 = H_1 = H_3 = 0,$$

we have (calling $M_2 = M$)

$$-4M = \frac{6 \times 30{,}000{,}000 \times 442}{120^2}(-2 \times \tfrac{1}{4} \text{ in.})$$

$$= 686{,}000 \text{ lb.-ins.}$$

To produce this moment in a 20 ft. simple span would require a load P determined by

$$M = \frac{PL'}{4}, \quad \text{i.e.,} \quad \frac{P \times 240}{4} = 686{,}000 \text{ lb.-ins.}$$

whence

$$P = 11{,}500 \text{ lbs.}$$

The unit stress is

$$S = \frac{Mc}{I} = \frac{686{,}000}{58.9} = 11{,}700 \text{ lbs. per sq. in.}$$

We may check the result from

$$= \frac{PL'^3}{48EI} = \frac{11{,}500 \times (240)^3}{48 \times 30{,}000{,}000 \times 442} = .25''.$$

This problem illustrates the fact that for short-span continuous beams a very high stress may result from a comparatively slight displacement of supports.

SECTION III.—CONTINUOUS AND SWING BRIDGES

74. General.—One of the most common cases of continuous girder action to be met with in bridge engineering practice is the swing bridge. This is a type of movable bridge which opens to admit the passage

of boats and barges by revolving in a horizontal plane on a central supporting pier. Such bridges are usually classed as *center bearing* and *rim bearing*. In the former the center reaction is carried entirely by a central pivot or its equivalent. (This may be a roller or disk bearing, but the statical effect is identical.) In the rim-bearing swing bridge the central support is a large circular girder upon which the main trusses or girders directly or indirectly rest. This circular girder or drum revolves with the bridge upon a set of conical rollers turning on a circular track. The diameter of this circular girder or drum varies with the span of the bridge; it may be as much as 25 ft. to 30 ft. In any case the main trusses or girders rest on two supports a considerable distance apart at the central pier, and we get in effect a continuous

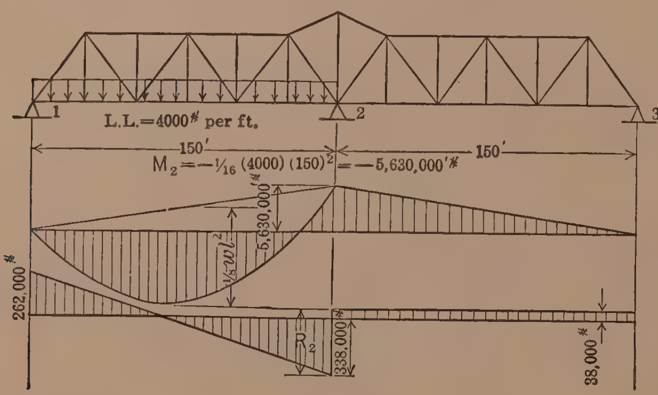


FIG. 104

girder of three spans. Figs. 104 and 105 show in outline a center bearing swing span and a rim-bearing span.

The center-bearing swing bridge is the simpler as to construction and operation and where feasible it will generally have the preference, though there is not complete unanimity of professional opinion on this point. The center-bearing type has been built for single and double track crossings up to a total length of 400 ft. and width *c.* to *c.* trusses of 40 ft. For spans up to 150 ft. the plate girder type is commonly used.

For extremely wide bridges the rim-bearing type is doubtless better suited, though it is also used frequently for single-track spans.

Swing bridge design is of itself a highly developed specialty in bridge engineering, and it is not proposed here to enter into any detailed discussion of the subject, other than that necessary to make clear the statical problem involved.

In contrast to most other bridges, the swing bridge is subjected to

loads under several different conditions. Thus when swinging it acts (under dead load alone) as a double cantilever; when closed the end supports are lifted (usually by means of wedges) an amount sufficient to prevent them from raising off the support under partial live load. This ordinarily means that we have a degree of continuous girder action for dead load with the span closed, and full continuous girder action for live load. But contingencies may arise when the wedges cannot be driven, in which case the end of the unloaded span will lift

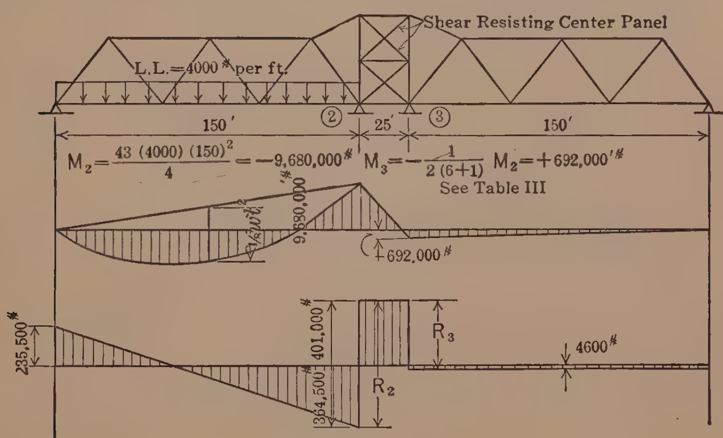


Fig. 105

entirely off the support under partial live loading, and the loaded span will act as a simple beam or truss.

The various combinations of stresses used in practical design will be indicated in the problem of Art. 76.

We are interested here primarily in the case where the bridge is closed and the ends raised so that full continuity of action may be assumed. We shall discuss the stress calculation for the center-bearing bridge and the rim-bearing bridge separately.

A. Center-bearing Swing Bridge

75. Method of Analysis.—A beam of uniform stiffness simply supported at three points on the same level is, as we have seen, one of the simplest of statically indeterminate problems. Calculations for a wide variety of cases show that practically any center-bearing swing bridge can be satisfactorily calculated on this basis, even though the span be a truss with considerable variation in depth. Two errors are involved in the process: (*a*) neglect of the varying moment of inertia, and (*b*) the omission of the effect of shear distortion. For trusses, both

these errors may be of considerable magnitude in themselves, but ordinarily they appear to be compensatory; in any case their effect on the final values of the redundant reaction (or moment) is slight.* A strictly accurate computation of the redundant reaction would involve the method of truss deflections as explained in Chapter II, problems (f) and (g), pages 105 and 106. There is general agreement among engineers that this is an unwarranted refinement, except in some very special cases.

Upon the foregoing assumption the analysis of the center-bearing swing bridge is fully illustrated in the problems of Figs. 104 and 106 The equation for the center moment is $-\frac{PL}{4}(k - k^3)$ (see page 177). If $P = 1$ this is the equation of the influence line for the center moment, and it may be easily constructed by the use of Table III, page 173.

If we have a distributed load extended from $k = k_1$ to $k = k_2$ we shall have

$$M = \frac{wL^2}{4}\int_{k_1}^{k_2}(k - k^3)dk = -\frac{wL^2}{8}[(k_2^2 - k_1^2) - \tfrac{1}{2}(k_2^4 - k_1^4)],$$

or we may use the tables of problem II, page 173.

Ordinarily broken loads are not considered in the design calculation for a center-bearing swing span, but the above equation is very useful in the special cases where broken loads need consideration.

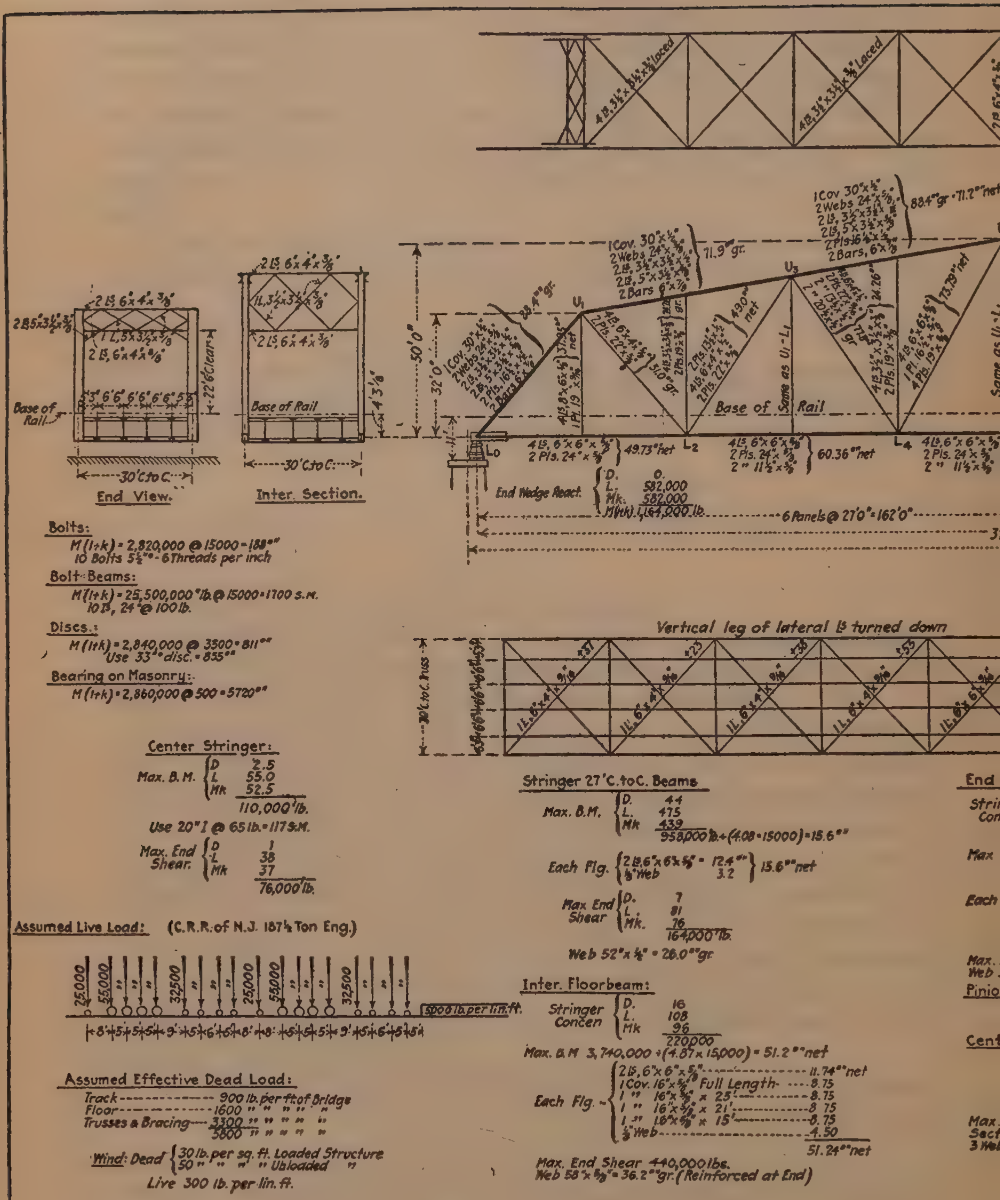
These will only occur when the bridge is so located (for example near a large switch yard) that it carries a great deal of mixed traffic. Even in such cases it is hardly reasonable to assume broken loads and full impact effect, since such broken up traffic would never occur at high speeds. If a reduced impact factor is used, broken loads, even if assumed as permissible, will rarely govern the design of a member.

If $k_1 = 0$, $k_2 = 1$, we have $M = -\frac{1}{16}wL^2$; if $k_1 = 0$, $k_2 = \frac{1}{2}$ (left half of left span loaded), we get $M = \frac{7}{256}wL^2$. For right half of same span loaded we must have $M = \frac{9}{256}wL^2$. The accompanying problem will illustrate fully the detail of the construction of influence lines for any particular member of the truss.

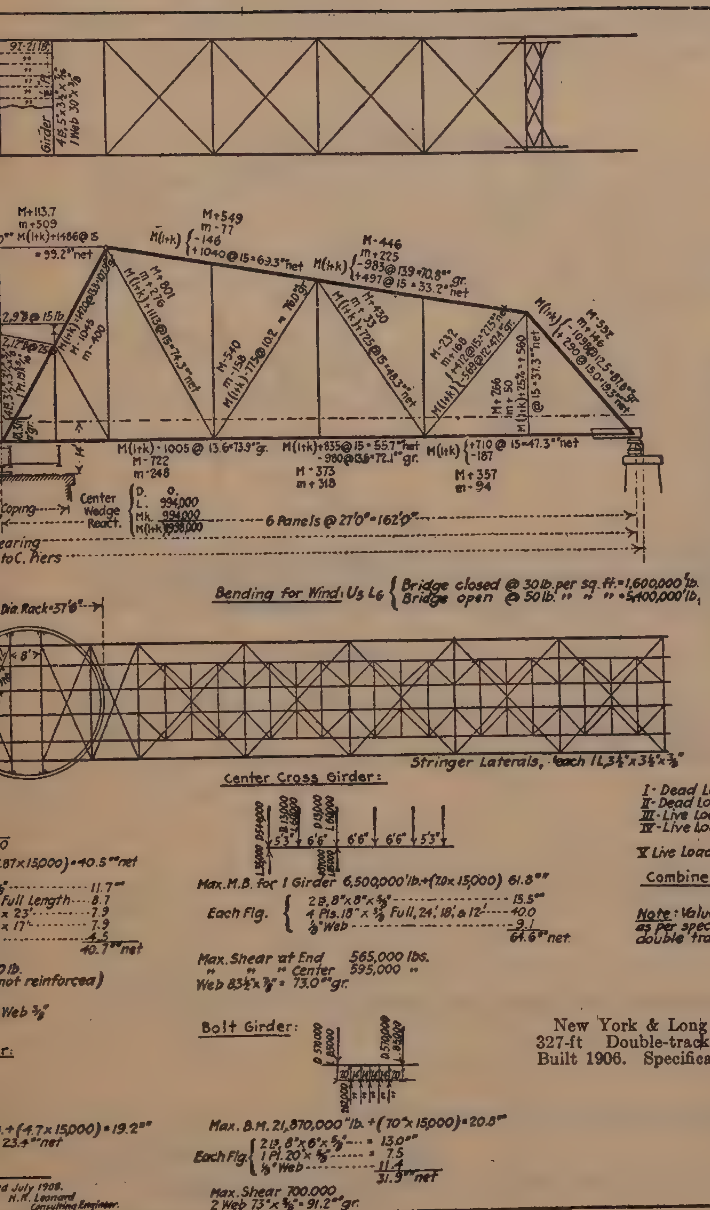
76. Example of Center-bearing Swing Bridge.—Fig. 106 is a stress sheet for a double track railway swing span.† The complete tabulation

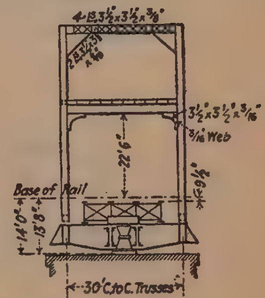
* See for example a comparative study of a typical case in Johnson, Bryan and Turneaure, "Modern Framed Structures," Part II, pp. 66–70. The maximum error in the bending moments at the joints was 1.5 per cent.

† This problem is one of a large collection of practical examples contained in the supplementary plates to F. C. Kunz's treatise on "Bridge Design." The solution presented there is considerably different from the above.



End View.
Inter. Section.
Base of Rail
30'C.to C.
22'6"Clear
2 Ls. 6"x4"x3/8"
50'0"
32'0"
U1
U3
L0
L2
L4
1 Cov. 30"x1/2"
2 Webs 24"x3/8"
2 Ls. 3 1/2"x3 1/2"x1/2"
2 Ls. 5"x3 1/2"x3/8"
2 Bars 6"x7/8"
71.9"gr.
88.4"gr.=71.2"net
Same as U1-L1
4 Ls. 6"x6"x5/8"
2 Pls. 24"x5/8"
49.73"net
60.36"net
End Wedge React.
D. 0.
L. 582,000
Mk. 582,000
M(I+k) 1,164,000 lb.
6 Panels @ 27'0"=162'0"
Bolts:
M(I+k)=2,820,000 @ 15000=188""
10 Bolts 5 1/2""-6 Threads per inch
Bolt Beams:
M(I+k)=25,500,000"lb. @ 15000=1700 s.m.
10 Ls, 24"@ 100 lb.
Discs.:
M(I+k)=2,840,000 @ 3500=811""
Use 33""disc.=855""
Bearing on Masonry:
M(I+k)=2,860,000 @ 500=5720""
Vertical leg of lateral Ls turned down
30'C.to C. Truss
Center Stringer:
Max. B.M. D 2.5, L 55.0, Mk 52.5, 110,000"lb.
Use 20" I @ 65 lb.=117 S.M.
Max. End Shear D 1, L 38, Mk 37, 76,000 lb.
Stringer 27'C.to C. Beams
Max. B.M. D. 44, L. 475, Mk 439, 958,000"lb.÷(4.08×15000)=15.6""
Each Flg. 2 Ls.6"x6"x5/8"=12.4"", 1/2" Web 3.2, 15.6""net
Max End Shear D. 7, L. 81, Mk. 76, 164,000 lb.
Web 52"x1/2"=26.0""gr.
Assumed Live Load: (C.R.R. of N.J. 187 1/2 Ton Eng.)
25,000 55,000 32500 25,000 55,000 32500
5000 lb. per lin. ft.
Inter. Floorbeam:
Stringer Concen. D. 16, L. 108, Mk 96, 220,000
Max. B.M 3,740,000÷(4.87×15,000)=51.2""net
Each Flg. 2 Ls, 6"x6"x5/8" 11.74""net
1 Cov. 16"x5/8" Full Length 8.75
1 " 16"x5/8" x 25' 8.75
1 " 16"x5/8" x 21' 8.75
1 " 16"x5/8" x 15' 8.75
1/8 Web 4.50
51.24""net
Max. End Shear 440,000 lbs.
Web 58"x5/8"=36.2""gr. (Reinforced at End)
Assumed Effective Dead Load:
Track 900 lb. per ft. of Bridge
Floor 1600 " " " "
Trusses & Bracing 3300 " " " "
5800 " " " "
Wind: Dead 30 lb. per sq. ft. Loaded Structure
50 " " " " Unloaded "
Live 300 lb. per lin. ft.





Section at Center.

Members	I	II	III	IV	V
U_1L_2	-136	-52	+293 -54	+220 -64	+108 -96
L_2U_3	+218	+141	+161 -155	-108 +182	+212
U_3L_4	-258	-196	+68 -268	-302 +38	+282
L_4U_5	+335	+276	+423 -16	+464	+466
L_0L_2	-39	+27	+385	+330 -55	+263 -55
L_2L_4	-246	-78	+520	+396 -127	+267 -127
L_4L_6	-548	-318	+246	+70 -174	-174
L_0U_1	+61	-42	-596	-510 +85	-407 +85
U_1U_3	+128	0	-523	-446 +97	-354 +97
U_3U_5	+394	+189	-420	-266 +155	-113 +155
U_5L_6	-498	-400	-520	-595	-547
U_5U_6	+184	+509	0	+209	+353
U_1L_1	+50	+50	+216	+216	+216
U_4L_4	-15				
U_6L_6	-40				

Stress noted in units of 1000 lb.
+ denotes tension.
− " compression.

Cases:

I - Dead Load Swinging
II - Dead Load Wedge Driven (Uplift 85,000 lb.)
III - Live Load Simple Span
IV - Live Load Cont. Span one arm loaded
V Live Load { Both arms loaded or one arm loaded & far end held down

Combine Cases: I & III, I & V, II & IV, II & V

Note: Values of k for main truss members, as per specifications. Multiplied by 0.8 for double track reduction.

New York & Long Branch R. R., Stress-sheet of 327-ft Double-track, Center-bearing Swing Span. Built 1906. Specifications P. R. R.

(To face page 182)

of stresses will be found at the right-hand side of the drawing, with the various legitimate combinations shown below. The following check calculations for three typical members, L_0U_1, U_3L_4 and U_3U_5 will indi-

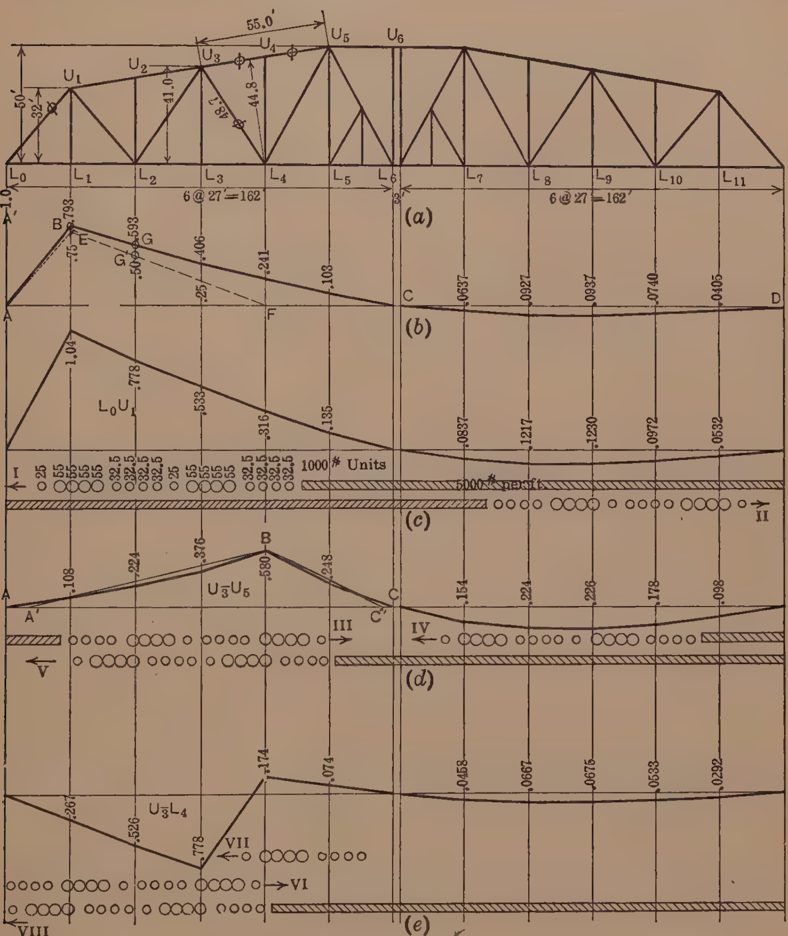


FIG. 107

cate the method of analyzing the statically indeterminate cases (IV and V in the figure) for such a truss.

77. *Influence Lines* (Fig. 107). From case (*a*), Art. 73, we have for P = unity and span = nL

$$M = -\frac{nL}{4}(k - k^3),$$

whence

$$R_1 = 1 - k - \tfrac{1}{4}(k - k^3),$$
$$R_2 = k + \tfrac{1}{2}(k - k^3) = \tfrac{3}{2}k - \tfrac{1}{2}k^3,$$
$$R_3 = -\tfrac{1}{4}(k - k^3).$$

Shear at left of center support $= -\frac{1}{4}(5k - k^3)$.
Shear at right of center support $= -\frac{1}{4}(k - k^3) = R_3$.

The influence line for R_1 is shown in Fig. 107*b*. From this the influence line for L_0U_1 is obtained by multiplying each ordinate by sec. $\theta = 1.31$ (Fig. 107*c*).

The influence line for U_3U_5 is also readily obtained by the aid of the influence line for R_1. The dotted line AEF in Fig. 107*b* indicates relatively the negative moment of the load unity when the latter is in the segment L_0L_4. For example the moment at L_4 due to unity at $L_2 = R_1 \times 4 \times 27 - 1 \times 2 \times 27 = 108\ (.593 - .50) = 108 \times GG'$ (Fig. 107*b*). Therefore if we take the difference of the ordinates between the diagrams ABC and $AEFC$ (as GG' at L_2) and multiply these by $\frac{108}{44.8}$, we shall get the corresponding ordinates to the stress influence line for U_3U_5 (Fig. 107*d*).

For U_3L_4 the effect of the upper chord slope is to *reduce* the vertical component of U_3L_4 by the vertical component of $U_3U_5 (= \frac{9}{55}U_3U_5)$ for all positions of the load to the right of L_4 and to *increase* it by a like amount for positions to the left of L_4. It is clear then that if we construct by means of the diagram for R_1 the shear influence line for the panel L_3L_4 and add algebraically to corresponding ordinates $\frac{9}{55}$ of the influence ordinates for U_3U_5, we shall obtain the graph for the vertical component of U_3L_4, and by multiplying by $\frac{48.7}{41.0}$ we get the final stress influence line for U_3L_4 as shown in Fig. 107*e*.

The calculations can best be carried out by a tabular scheme as follows:

	L_1	2	3	4	5	7	8	9	10	11
(1) 9–55 × U_3U_5	.0177	.0366	.0618	.0947	.0405	− .0251	− .0365	− .0368	− .0291	− .0160
(2) Shear	− .207	− .407	− .594	.2410	.1030	− .0637	− .0927	− .0937	− .0740	− .0405
Subtract (1) from (2)	− .2247	− .4436	− .6558	.1463	.0625	− .0386	− .0562	− .0569	− .0449	− .0245
Times 48.7 ÷ 41.0	− .267	− .526	− .778	.174	.074	− .0458	− .0667	− .0675	− .0533	− .0292

The influence line for any member of the truss may be drawn similarly to one of the above.

78. *Live Load Stresses.*—It will be observed that the influence lines for the preceding cases in general change slope at each panel point. It is not feasible in such cases to develop useful algebraic criteria for the position of a train of wheel concentrations giving maximum stresses, as is done for the case of simple trusses. The maximum values are best obtained by repeated trial, guided by the general principle that the heavier loads should be placed in the region of large ordinates. Figs. 107 (*c*), (*d*), and (*e*) show the correct positions for maximum loadings for the various cases obtained in this way. The results are perhaps most rapidly obtained if the influence line is drawn to a fairly large scale and the ordinates corresponding to the loads scaled from this, grouping the loads and taking the ordinate through the center of gravity where possible. The work may be carried out readily enough arithmetically, however, as illustrated in the following calculation of the stress in $L_0 - U_1$ for position (*I*).

$$
\begin{array}{lr}
25 \quad \times \frac{14}{27}(1.04) & = 13.7 \\
55 \quad \times \frac{22}{27}(1.04) & = 46.9 \\
3 \times 55 \quad \times (.778 + \frac{21}{27} \times .262) & = 164.0 \\
4 \times 32.5 \times .778 & = 101.1 \\
25 \quad \times (.553 + \frac{11}{27} \times .245) & = 15.8 \\
4 \times 55 \quad \times (.316 + \frac{22}{27} \times .217) & = 108.7 \\
4 \times 32.5 \times (.135 + \frac{25}{27} \times .181) & = 39.7 \\
5 \times 12.5 \times (.135 + \frac{6}{27} \times .181) & = 10.9 \\
5 \times 27.0 \times .0675 & = 9.2 \\
\hline
 & 510.0
\end{array}
$$

This gives the maximum stress in L_0U_1 due to *both* tracks loaded, on left arm only, as indicated in Fig. 107 (*c*) — (*I*).

If we assume the near track loaded as in (*I*) but with the uniform load extending over the right arm, and the far track loaded with the uniform train load in both arms, we have:

(*a*) Stress in L_0U_1 from load on near track

$$= \frac{21.5}{30}(-510 + 5.0 \times \text{influence area for right arm})$$

$$= (-510 + 5.0 \times 13.0) \times \frac{21.5}{30} = -445 \times \frac{21.5}{30} = -318.5.$$

(*b*) Stress in L_0U_1 due to load on far track

$$= \frac{8.5}{30}(5.0 \times \text{difference in influence areas for right and left spans}),$$

$$= \frac{8.5}{30}[5.0(-75.7 + 13.0)] = -88.5,$$

whence final stress in L_0U_1 for this case $= -407$.

The maximum stresses for U_3U_5 and U_3L_4 (shown in table of Fig. 106) are similarly obtained.

79. *Approximate Criteria for Maximum Wheel Loading.*—It is worth noting that while the influence lines are irregular figures to which simple algebraic criteria cannot be applied, nevertheless, in the regions where the loads have the greatest effect (the left-hand segments in Figs. 107 (c), (d) and (e), the form of the influence line approaches more or less roughly that of a triangle. This fact may aid the computor in determining the character of the trial loadings. Applying the ordinary criterion for the triangular influence line

$$W\frac{l_1}{l} \begin{matrix} > W_1 \\ < W_1 + P_c \end{matrix}$$

where W = total load,
W_1 = load on short segment of influence line,
l = total span covered by influence line,
l_1 = length of short segment of influence line,
P_c = critical wheel,

we find that loadings (I), (V), (VII), and (VIII) satisfy this criterion, while loadings (III) and (VI) vary by only one wheel space from the position indicated, and almost the same numerical result would have been obtained by using the position indicated by a triangular influence line.

80. *Equivalent Uniform Loads.*—The above result suggests the possibility of using an equivalent uniform load * to obtain the live load stresses. The specified loading is of a special type approximating Cooper's E-50, differing only in the driver axle loads, which are 10 per cent heavier. For loaded lengths used in the left span of the truss the total excess of the given load over E-50 is about 5 per cent. For an influence line such as U_3L_4 the predominant effect of the heavy drivers would justify selecting an equivalent load of perhaps 7 or 8 per cent in excess of E-50.

Making these tentative and rather crude approximations, we get

* It is presumed that the student is familiar with the general subject of equivalent uniform live loadings from his previous study of bridge analysis. Those who are not familiar with the subject and those who wish to study it further are referred to "Live Load Stresses in Railway Bridges," by George E. Beggs; "Modern Framed Structures," Part I, by Johnson, Bryan and Turneaure, and "Live Loads for Railway Bridges," by D. B. Steinman, Transactions of American Society of Civil Engineers, Vol. LXXXVI. Professor Beggs' treatise contains elaborate tables, and Professor Turneaure's book and Dr. Steinman's article contain convenient charts of equivalent live loads. The chart shown in Fig. 107f is practically identical with that given in "Modern Framed Structures."

the following typical results for the equivalent live load check on the previous figures.

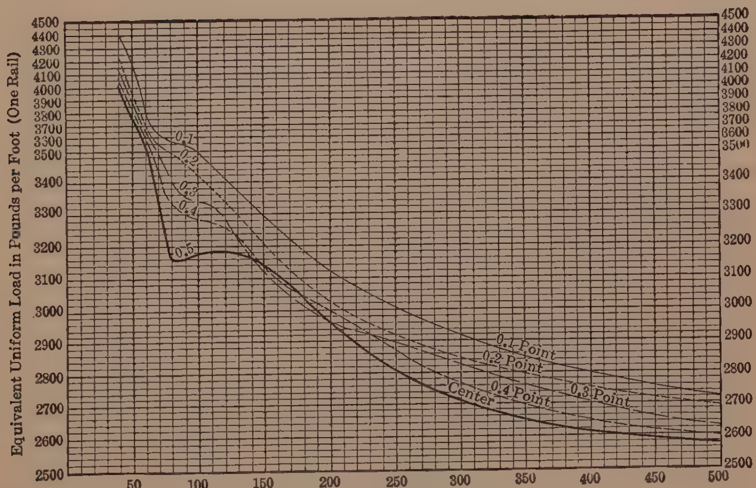
For L_0U_1 . . . Equivalent uniform load for E-50 (see Fig. 107f) = $\frac{1}{6}$ point in 162 ft. span = 6440

$$6440 \times 1.05 = 6750;$$

area of influence line = 75.7

$$75.7 \times 6750 = 512,$$

as against 510 by wheel load calculation.



Equivalent Uniform Loads for Cooper's E-50 Loading

FIG. 107f

For U_3U_5, Equivalent uniform load for E-50 ($\frac{1}{3}$ point in 162-ft. span = 6160

$$6160 \times 1.05 = 6470,$$

area of influence line = 41.4

$$6470 \times 41.4 = 268.0,$$

as against 266 by wheel load calculation.

For U_3L_4, Equivalent uniform load for E-50 ($\frac{2}{10}$ point in 103-ft. span) = 6800.

Here the effect of the heavy drivers might be expected to have a larger influence on the result than would be indicated by the ratio of the total weight of loading to total E-50 loading of same length. It

seems fair to assume the added equivalent load per foot to be 8 per cent rather than 5 per cent as previously used on the longer spans.

Area of influence line = 40.6.

$$6800 \times 1.08 \times 40.6 = 299,$$

against 302 by wheel load calculation.

Using an increase of 5 per cent as in the other cases gives 290, or an error of less than 4 per cent.

These calculations tend to show that for all ordinary cases, calculation by the equivalent uniform load method gives results which are quite as accurate as the data justify, and where tables or graphs of equivalents are available, it is the method recommended.

For such an influence line as that of Fig. 107*d*, slightly closer results in selecting the equivalent load may be obtained by using the triangular influence line of equal area . . . $A'BC'$. This correction is unnecessary except for such cases as deviate considerably from the triangular form.* Where the influence line does not even approximate a triangular form (as in right-hand portion of Fig. 107*c*) the above method can be regarded only as a very rough approximation, if applicable at all.

B. Rim-bearing Swing Bridge

81. General Considerations.—We have just seen that the ordinary continuous girder theory applies with sufficient accuracy to the center-bearing swing bridge. We shall find on the contrary that important modifications in the analysis must be made before it can be applied to the rim-bearing swing bridge. In the 3-span girder with the center span very much shorter than the others, the shearing distortion cannot be ignored without introducing serious error; the deflection in this short panel due to shear is, as a matter of fact, of the same order of magnitude as the moment deflection. No practical bracing is possible, even if desirable, which is stiff enough to minimize the shearing deflection to such extent that the usual flexural theory may be applied. But it is generally conceded that the use of such bracing is undesirable; the high shearing stress and large negative reaction at the intermediate support adjacent to the unloaded span introduce serious practical difficulties. It may be shown that if we calculate the reactions on the basis of the continuous girder theory, and proportion the web members of the center span to carry the large shear indicated by this theory, and

* For further discussion on this general subject, see Johnson, Bryan and Turneaure's "Modern Framed Structures," Part II, page 65, and Charles A. Ellis, "Essentials of Structures," page 309 et seq.

then make an exact calculation by the truss-deflection method, we shall find that no such large shear in the center span is actually realized. That is to say, though the bracing is *strong* enough to carry the large shear, it is not *stiff* enough to cause it to develop. Indeed, within

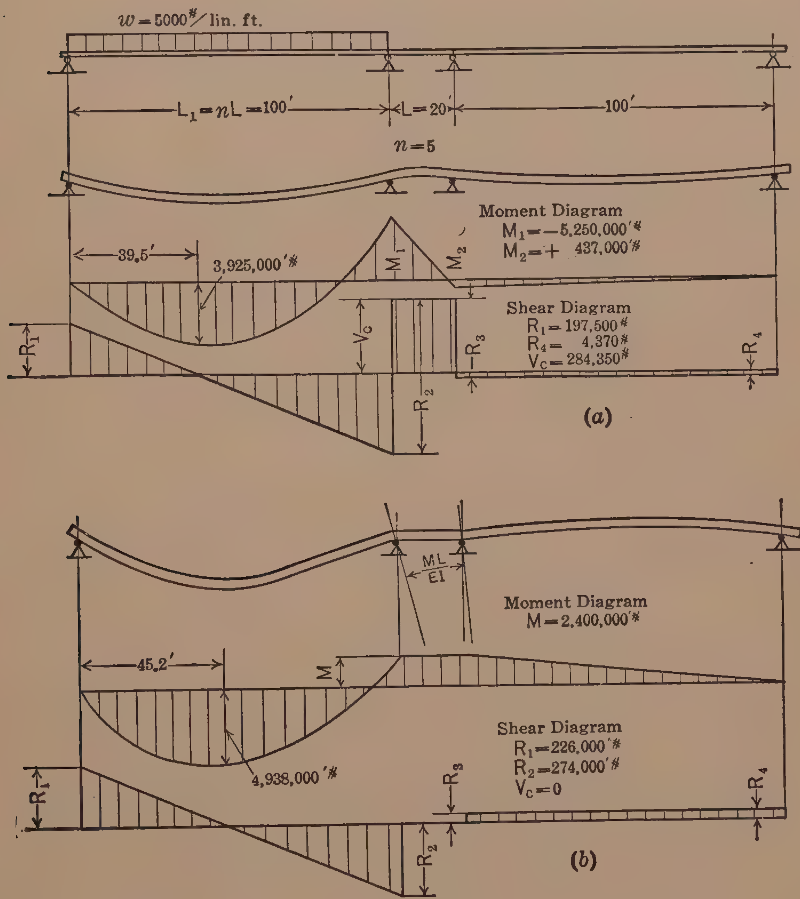


FIG. 108

reasonable limits, the size of the bracing appears to have little effect on the unit stress.*

These facts have led to the general adoption of very light center panel bracing, just large enough to stiffen the structure properly but assumed

* See Johnson, Bryan and Turneaure, "Modern Framed Structures," Part II, p. 79.

not to transmit any shear. Such a truss is said to be *partially continuous*.

Figs. 108*a* and 108*b* illustrate the essential features of the two types of action.

82. Equations for Shear and Moment—Full Continuity.—To develop in a more convenient form the expressions for shear and moment, we may write the three-moment equation, assuming $M_1 = M_4 = 0$, $L =$ center span; $nL =$ outer spans, load P in left span,

$$2M_2(nL + L) - M_3L = Pn^2L^2(k - k^3),$$

and

$$-M_2L - 2M_3(nL + L) = 0,$$

whence

$$M_2 = -PnL \cdot (k - k^3)\frac{2n^2 + 2n}{4n^2 + 8n + 3},$$

$$M_3 = +PnL(k - k^3)\frac{n}{4n^2 + 8n + 3}.$$

Shear in center panel

$$V_c = \frac{M_2 - M_3}{L} = P(k - k^3)\frac{2n^3 + 3n^2}{4n^2 + 8n + 3}.$$

For uniform load over left span,

$$M_2 = \frac{wn^2L^2}{4}\,\frac{2n^2 + 2n}{4n^2 + 8n + 3}.$$

Tables III, VII and VII*b* may be used to evaluate terms involving the constants $k - k^3$ and $\dfrac{2n^2 + 2n}{4n^2 + 8n + 3}$.

Fig. 105 shows a numerical case worked out for uniform load.

It will be noted that the shear in the center panel is 401,000# and the negative reaction at R_3 = 405,600#. It would be very difficult if not impossible to provide satisfactorily for the uplift that would occur at this point under combined dead and partial live loading, and as previously noted, both experience and a more exact analysis indicate that even with bracing heavy enough to carry it safely, no such shear is actually developed, nor anything approaching it, and the ordinary continuous girder theory thus appears quite inapplicable to the rim-bearing swing bridge.

83. Theory of Partially Continuous Truss.—The assumption of partial continuity—i.e., that the shear in the center panel is zero, gives results which are in fair agreement with the more exact analysis even

when the center span is heavily braced, and for light bracing the agreement is quite satisfactory.

The corresponding formulae for the moments over the center supports (if the shear in the panel is zero, these moments must be equal)

TABLE VII

Rim-bearing Swing Bridge Shear in Center Panel

From equations in Art. 82:

$$-2M_1(L_1 + L) - M_2L = PL_1^2(k - k^2)$$
$$-2M_2(L_1 + L) - M_1L = 0$$

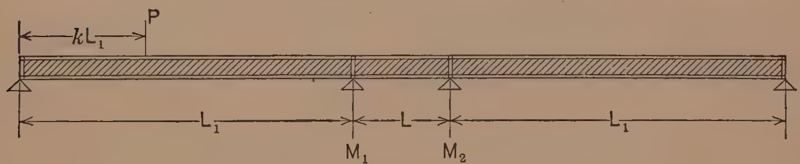
Let $L_1 = nL$.

Then:

$$-2M_1(1 + n) - M_2 = n(k - k^3)PL_1$$
$$-M_1 - 2M_2(1 + n) = 0.$$

Solving

$$M_1 = \left[-\left(\frac{2n(n+1)}{4n^2 + 8n + 3}\right)(k - k^3)\right]PL_1.$$



Values of $\left[\left(\frac{2n(n+1)}{4n^2 + 8n + 3}\right)(k - k^3)\right]$

Values of k	$n = 20$	$n = 10$	$n = 8$	$n = 6$	$n = 5$	$n = 4$	$n = 3$	$n = 2$	$n = 1$
.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
.05	.0238	.0227	.0223	.0215	.0210	.0202	.0191	.0172	.0134
.10	.0471	.0450	.0441	.0426	.0416	.0400	.0377	.0339	.0264
.15	.0700	.0669	.0655	.0632	.0617	.0594	.0561	.0504	.0393
.20	.0914	.0873	.0853	.0825	.0806	.0776	.0732	.0658	.0513
.25	.1112	.1064	.1041	.1008	.0982	.0945	.0892	.0803	.0625
.30	.1298	.1242	.1214	.1174	.1145	.1103	.1038	.0936	.0729
.35	.1458	.1398	.1364	.1320	.1290	.1240	.1171	.1053	.0820
.40	.1598	.1528	.1493	.1445	.1411	.1357	.1280	.1152	.0897
.45	.1708	.1633	.1596	.1545	.1510	.1450	.1370	.1232	.0958
.50	.1783	.1708	.1670	.1612	.1575	.1514	.1430	.1286	.1002
.55	.1825	.1745	.1709	.1650	.1613	.1550	.1464	.1318	.1026
.60	.1825	.1745	.1709	.1650	.1613	.1550	.1464	.1318	.1026
.65	.1783	.1708	.1670	.1612	.1575	.1514	.1430	.1286	.1002
.70	.1698	.1625	.1589	.1534	.1498	.1441	.1359	.1225	.0953
.75	.1560	.1492	.1460	.1409	.1377	.1325	.1250	.1124	.0876
.80	.1370	.1310	.1282	.1237	.1208	.1165	.1097	.0988	.0768
.85	.1124	.1073	.1050	.1016	.0992	.0954	.0900	.0809	.0630
.90	.0814	.0777	.0762	.0737	.0718	.0691	.0654	.0586	.0456
.95	.0442	.0423	.0414	.0400	.0391	.0375	.0354	.0319	.0248
1.00	0.0000	0.0000	.0000	0.0000	0.0000	.0000	0.0000	0.0000	0.0000

may be developed conveniently by the principle of moment areas. We may consider the two center supports coinciding and the action the same as in a two-span girder except that there will be a break in continuity over this center support equal to the angle between the two verticals at the inner ends of the two spans. This angle will evidently be measured by the sum of the top and bottom chord deformations divided by the height of the truss, that is

$$\alpha = \frac{\Delta L}{h} = \frac{sL}{AEh} = \frac{ML}{h^2AE} = \frac{ML}{EI},$$

and

$$EI\alpha = ML. \qquad (a)$$

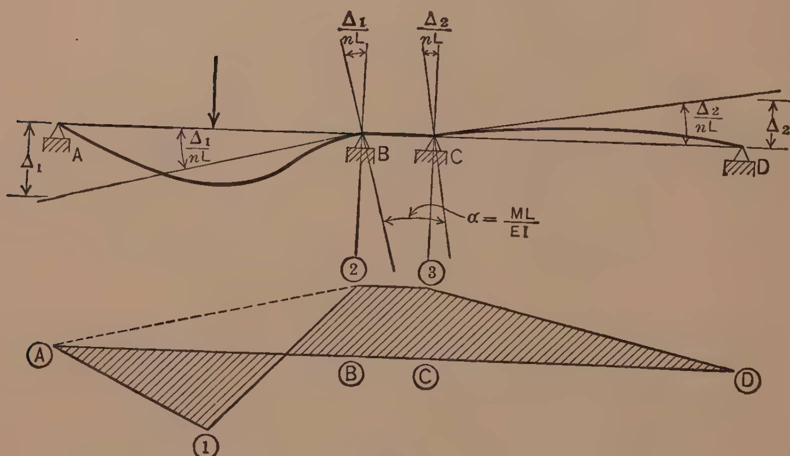


FIG. 109

We have from Fig. 109

$$\text{Moment of area } A\text{-}1\text{-}2 \text{ about } A = \frac{P}{6}n^3L^3(k - k^3).$$

$$\text{Moment of area } A\text{-}2\text{-}B \text{ about } A = \frac{n^2L^2M}{3}.$$

$$\therefore\ EI\Delta_1 = \frac{L^2}{6}[n^3PL(k - k^3) - 2n^2M]. \qquad (b)$$

$$\text{Moment of area } 3\text{-}C\text{-}D \text{ about } D = \frac{n^2L^2M}{3} = EI\Delta_2. \qquad (c)$$

Now it is clear from the figure that

$$\frac{\Delta_2}{nL} + \alpha = \frac{\Delta_1}{nL},$$

and if we multiply through by EI and substitute the values from (a), (b), and (c), we have

$$L\left[\frac{n^2PL(k - k^3) - 2nM}{6}\right] = ML\left(1 + \frac{n}{3}\right),$$

whence

$$M = -PnL(k - k^3)\frac{n}{4n + 6}. \quad . \quad . \quad . \quad . \quad (39)$$

Fig. 108*b* shows the large reduction in the moment at the center support as compared with the continuous girder calculation.

For calculation of moments and shears or of influence lines, it will generally be convenient to use the reactions R_1 and R_4. For a load on the left span these may be written

$$R_1 = P\left[1 - k - (k - k^3)\frac{n}{4n + 6}\right] = P(1 - 2k + k^3)\frac{n}{4n + 6}. \quad . \quad (40)$$

$$R_4 = -P(k - k^3)\frac{n}{4n + 6}. \quad . \quad . \quad (41)$$

Tables III and VII*a* will aid in evaluating these expressions numerically.

TABLE VII*a*

Values of $\frac{n}{4n + 6}$

n	1	2	3	4	5	6	7	8	9	10	15	20
$\frac{n}{4n + 6}$	.100	.143	.167	.182	.192	.200	.206	.211	.214	.218	.227	.232

TABLE VII*b*

Values of $\frac{2n^2 + 2n}{4n^2 + 8n + 3}$

n	1	2	3	4	5	6	7	8	9	10	15	20
$\frac{2n^2 + 2n}{4n^2 + 8n + 3}$	.267	.343	.381	.404	.420	.430	.438	.445	.451	.455	.469	.476

84. Example of Rim-bearing Swing Bridge.—To illustrate the analysis of the partially continuous rim-bearing swing span we shall show the influence lines for a rim-bearing span identical with the center-bearing span of Fig. 107, except for the addition of a small center panel of 16′ (see Fig. 110).

The equations of the preceding article enable us to compute all influence ordinates in a manner similar to the example for the center-

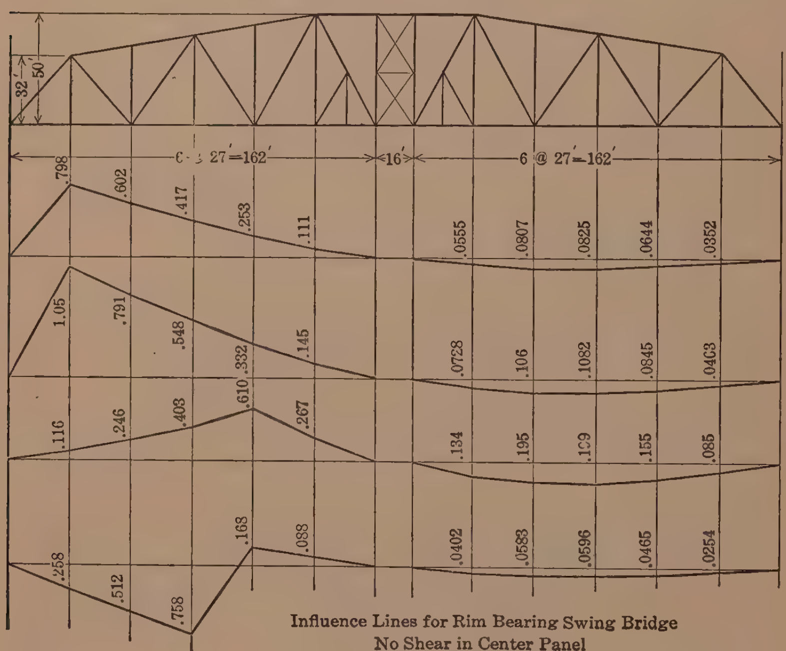


FIG. 110

bearing bridge. The results are shown in the figure. A comparison with the influence lines of Fig. 107 shows that while some of the smaller ordinates differ by as much as 15 to 20 per cent, the larger ordinates rarely differ by more than 4 or 5 per cent. On this account it is not uncommon to analyze the rim-bearing swing bridge as a center-bearing bridge with spans equal to the two outside spans. The variations in the actual sections for the two cases will usually be of no practical importance.

C. Present Status of Swing Bridge

84a. As recently as thirty years ago the swing bridge was the prevailing type of movable bridge in America. Swing spans have been built ranging from 100-ft. plate girders to trusses more than 500 ft. in length. Their relative popularity has greatly decreased of recent years, due to the remarkable developments in the design of the bascule and vertical lift types. These latter are ordinarily statically determinate types and hence are not treated in this book, nor would it be in place here to enter into a discussion of the relative merits of the various types of movable bridges. It may be noted that both the bascule and vertical lift types have the advantage that they may be opened more quickly than swing spans and afford an unobstructed waterway. Double leaf bascule bridges have been built up to 350-ft. spans and vertical lift bridges up to 450 ft. In spite of the increasing frequency of such types, however, the swing bridge still appears to have a very definite and important place in movable bridge design, particularly for longer spans, and hence it is felt that the space given to it in this chapter is justified.

D. Turntables

84b. The locomotive turntable is a structure of common occurence which is built sometimes as a simple girder and sometimes as a continuous girder. The latter type presents a problem statically identical with the center-bearing swing bridge, and from the standpoint of analysis no separate treatment is required.

E. Continuous Bridges

84c. Until very recently the Lachine bridge of the C. P. R. Ry. over the St. Lawrence river was practically the only continuous truss span (aside from swing bridges) in America. With the completion of the Sciotoville bridge in 1917, however, the interest in this type of bridge has greatly increased, and there are now several other large continuous structures completed or projected.

It has always been recognized that the continuous truss possesses, in some respects, marked advantages over a corresponding construction involving two or more simple spans. In general for the same loading the stresses are lower, the structure is more rigid, construction details are to some extent simplified, and where cantilever erection is necessary or desirable, the continuous truss is far better adapted to it.

Two objections have been largely responsible for the infrequency of its use:

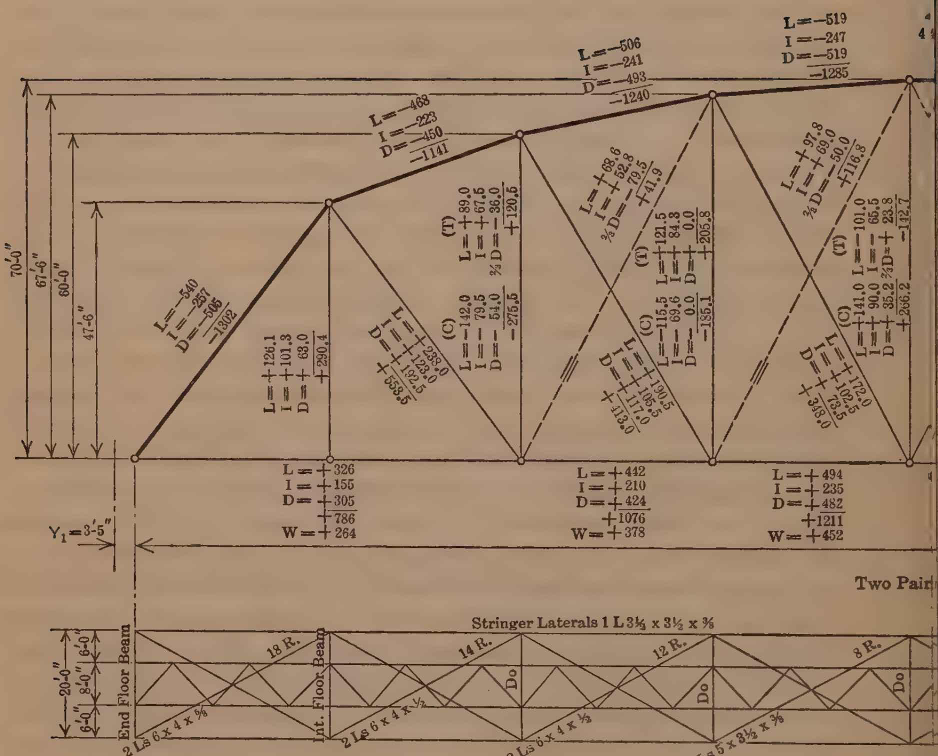
(*a*) Where the live load is large there will be marked reversals of stress in the chord members, requiring, according to most standard specifications, much lower unit stresses, which tends largely to offset the economy otherwise secured, and

(*b*) Small relative changes in the levels of the supports were believed to seriously disturb the normal stress conditions. Experience and more thoroughgoing investigation have shown that the second objection has little weight in the case of large structures * (to which the continuous type is especially adapted), while the matter of stress reversal is of diminishing importance the longer the span and hence the greater relative importance of the dead load. Further, there is an apparently growing opinion among structural engineers that the severe penalties placed upon alternating stresses in the past are without justification in fact. These are among the considerations which have led to the present increase in favor of the continuous bridge; this favorable opinion seems to be rapidly growing and this type of construction will doubtless be widely used in the future for long-span bridges.

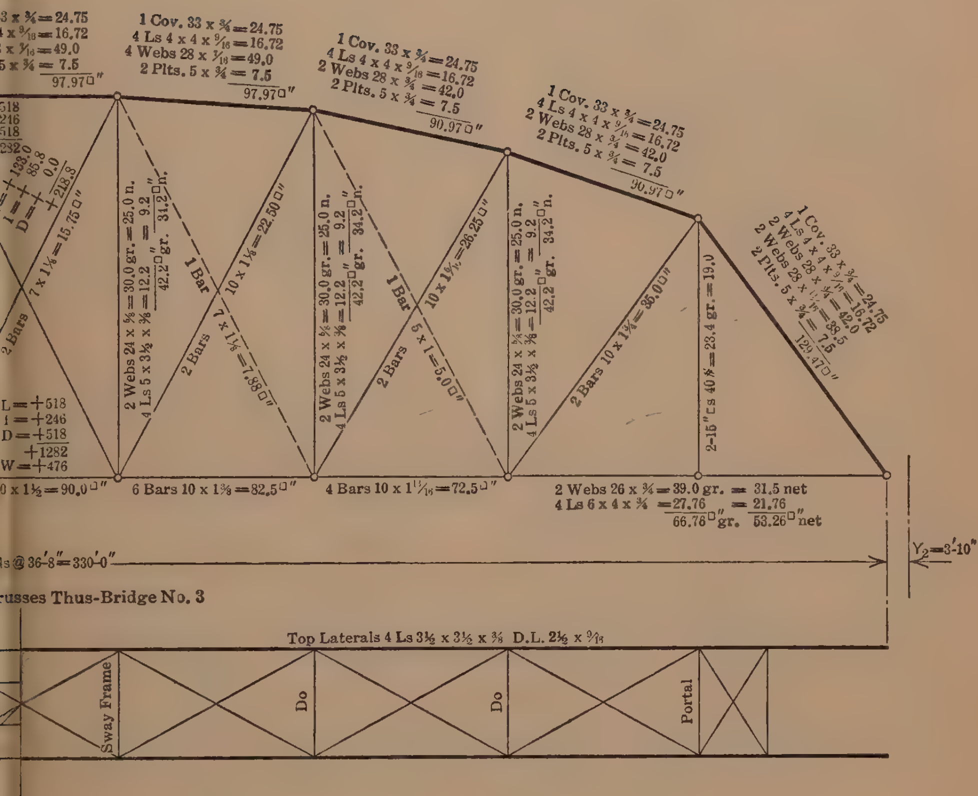
Like the turntable, the two-span continuous bridge is statically identical with the center-bearing swing span when the latter is closed and the end supports in full action. The three-span continuous truss, on the other hand, unlike the three-span (rim-bearing) swing bridge, may be analyzed to a close approximation by the ordinary continuous girder theory. The three spans are usually of somewhere near equal length and hence we do not have the exaggerated importance of the shearing deflections which arise from the relatively very short center span of the rim-bearing swing bridge. It will be clear from this that no new principles, and practically no change in method of application are required for the analysis of two- and three-span continuous bridges (the only common types). However, the following example will be of interest in showing a comparative design of a continuous truss with two simple spans in a "border line" case, i.e. about the minimum span limit for which continuous construction would be expected to compete with simple trusses.

84d. Example.—In Fig. 110*a* is shown a stress sheet together with the makeup of all members for a 330-ft. Pratt type simple truss span. It is a single track pin-connected bridge designed according to the 1910 A.R.E.A. specifications. This particular crossing consisted of two consecutive 330-ft. spans, which gave the opportunity of figuring an alter-

* See Chapter VIII for further discussion of this point.



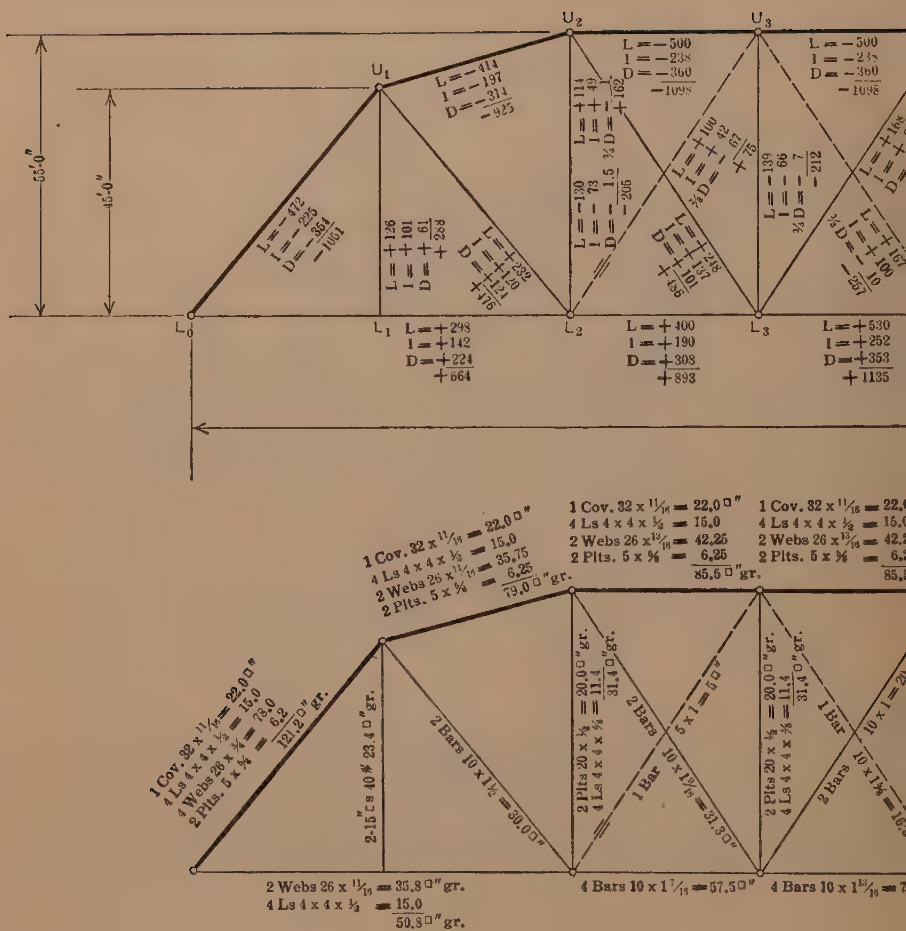
L = −519
I = −247
D = −519
−1285
L = −506
I = −241
D = −493
−1240
L = −468
I = −223
D = −450
−1141
L = −540
I = −257
D = −505
−1302
L = +126.1
I = +101.3
D = + 63.0
+290.4
L = +238.0
I = +128.0
D = +192.5
+558.5
(T)
L = + 89.0
I = + 67.5
⅔ D = − 36.0
+120.5
(C)
L = −142.0
I = − 79.5
D = − 54.0
−275.5
L = + 68.6
I = + 52.8
⅔ D = − 79.5
+ 41.9
(T)
L = +121.5
I = + 84.3
D = + 0.0
+205.8
(C)
L = −115.5
I = − 69.6
D = − 0.0
−185.1
L = +190.5
I = +105.5
D = +117.0
+413.0
L = + 97.8
I = + 69.0
⅔ D = − 50.0
+116.8
(T)
L = −101.0
I = − 65.5
⅔ D = + 23.8
−142.7
(C)
L = +141.0
I = + 90.0
D = + 35.2
+266.2
L = +172.0
I = +102.5
D = + 73.5
+348.0
70′-0″
67′-6″
60′-0″
47′-6″
L = + 326
I = + 155
D = + 305
+ 786
W = + 264
L = + 442
I = + 210
D = + 424
+1076
W = + 378
L = + 494
I = + 235
D = + 482
+1211
W = + 452
Y₁ = 3′-5″
Two Pair
Stringer Laterals 1 L 3½ x 3½ x ⅜
18 R.
14 R.
12 R.
8 R.
Do
Do
Do
20′-0″
6′-0″
8′-0″
6′-0″
End Floor Beam
Int. Floor Beam
2 Ls 6 x 4 x ⅝
2 Ls 6 x 4 x ½
2 Ls 6 x 4 x ½
2 Ls 5 x 3½ x ⅜



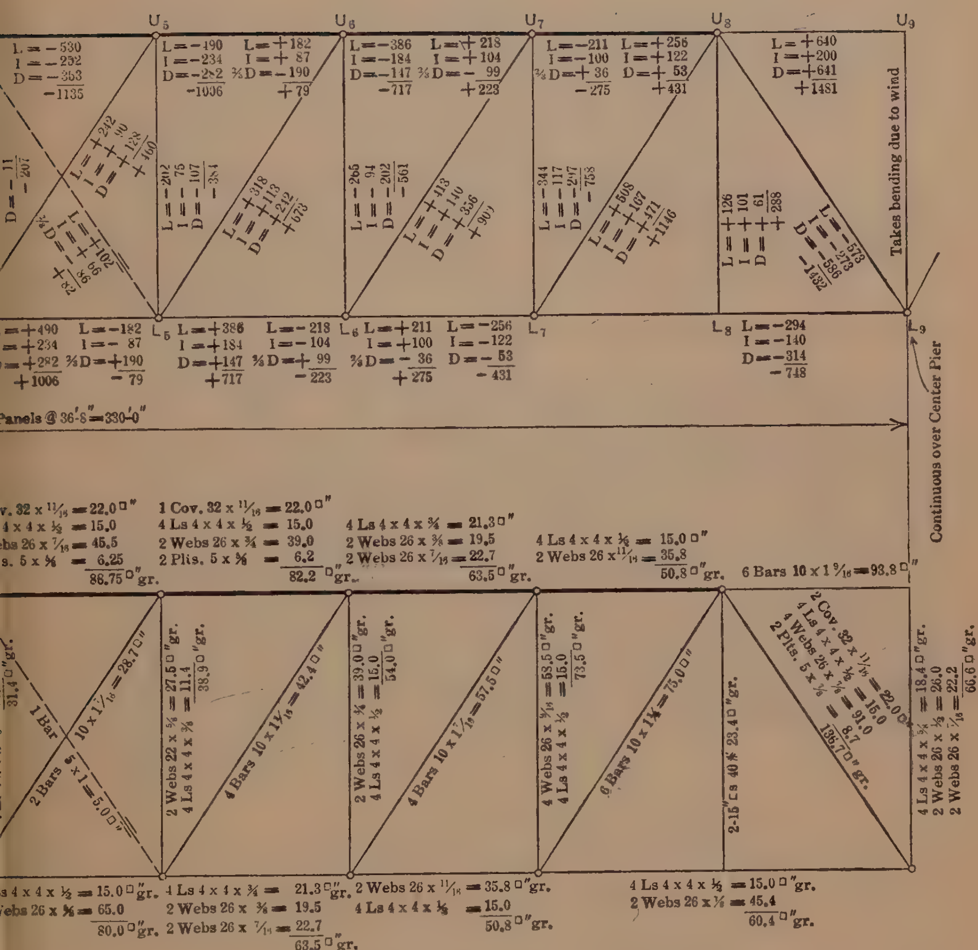
usses Thus-Bridge No. 3

110a

(*To face page 196*)



U1
U2
U3
L0
L1
L2
L3
55'-0"
45'-0"
L = -500
I = -238
D = -360
-1098
L = +298
I = +142
D = +224
+664
L = +400
I = +190
D = +303
+893
L = +530
I = +252
D = +353
+1135
1 Cov. 32 x 11/16 = 22.0□"
4 Ls 4 x 4 x ½ = 15.0
2 Webs 26 x 11/16 = 35.75
2 Plts. 5 x ⅝ = 6.25
79.0□"gr.
1 Cov. 32 x 11/16 = 22.0□"
4 Ls 4 x 4 x ½ = 15.0
2 Webs 26 x 13/16 = 42.25
2 Plts. 5 x ⅝ = 6.25
85.5□"gr.
2 Webs 26 x 11/16 = 35.8□"gr.
4 Ls 4 x 4 x ½ = 15.0
50.8□"gr.
4 Bars 10 x 1 7/16 = 57.5□"
2 Bars 10 x 1½ = 30.0□"



(*To face page 196*)

nate in which the two consecutive simple spans were made continuous.

Certain limitations imposed by the A.R.E.A. specifications made it impossible to use the same depth for the continuous bridge as for the two simple spans. The "thickness to supported width" ratios of $\frac{1}{30}$ for web plates and $\frac{1}{40}$ for cover plates of compression members together with the requirement of clearance for I-bar heads, and also the requirement of riveter clearance between the inturned flanges of the vertical posts, were the most important of these limitations.

The problem of obtaining a section for the top chords and end posts which would satisfy these limitations, required that a shallower depth be used for the continuous bridge. Also the effect of a curved topchord in reducing web-stresses was missing at one end of each of the two continuous spans.

The stress sheet together with the makeup of members for the continuous bridge is shown in Fig. 110*b*.

The weight of the continuous bridge (assuming the same floor system and bracing for each type) was almost exactly the same as that for the two simple spans. It is believed that if requirements for compression members were less rigid so that a deeper truss could have been used for the continuous bridge, a small percentage of saving would have resulted. This fact has of course already been established for bridges of longer spans and heavier loads.

In Fig. 110*c* is shown the method of calculating live-load stresses for the continuous bridge. Dead-load stresses were calculated independently of the live-load stresses. The dead panel load equals (2600) (36.67) = 95,400# and the reaction at the non-continuous end is obtained from the equation for R_1 in Art. 77,

$$R = P\left[(1 - k) - \left(\frac{k - k^3}{4}\right)\right],$$

where $k = \frac{1}{8}$ for L_1, $\frac{2}{8}$ for L_2, etc. The product of

$$95400 \times \Sigma\left[(1 - k) - \left(\frac{k - k^3}{4}\right)\right],$$

for the span considered, minus $95400 \times 2\frac{k - k^3}{4}$ for the other span gave a reaction at L_0 of 275,000#.

The calculation of all live-load stresses was based entirely upon the influence line for the reaction at L_0. These values are shown in Fig. 113 as the ordinates to the line 1–15–18–30 measured from the base line

LIVE LOAD STRESSES IN A CONTINUOUS TRUSS OF 330′ SPANS BY THE METHOD OF INFLUENCE LINES

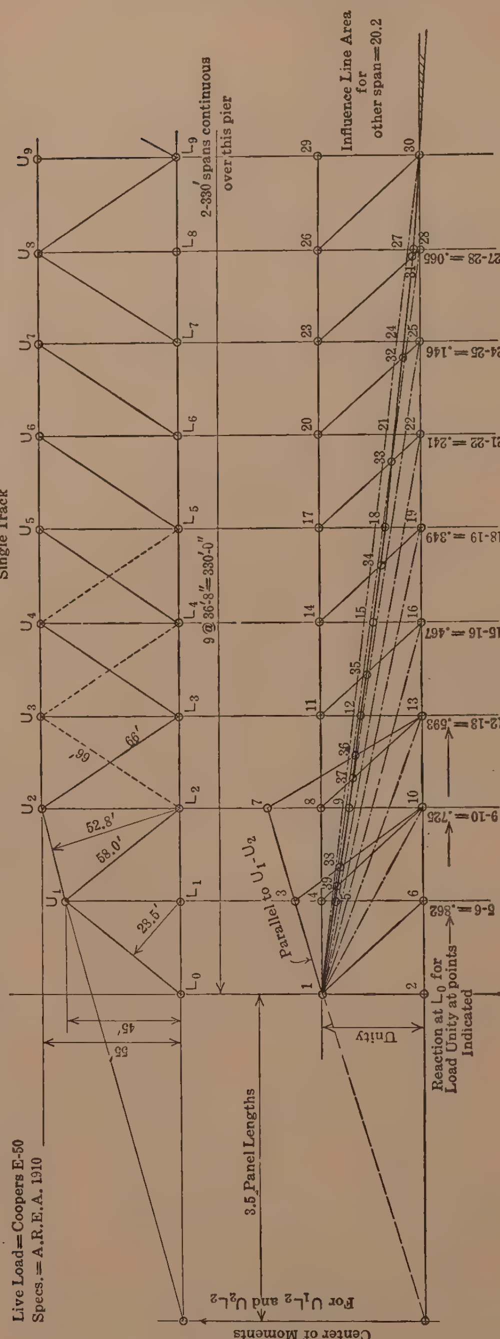


FIG. 110 (c)

Chord Members

Area for chord members equals amount which (for unit load per linear foot) is multiplied by distance from L_0 to any panel point to get the moment at that point.

Factor equals distance (defined under *area*) divided by moment arm of chord. L.L. *Stress* equals product of area times factor times equivalent uniform L.L.

	L_0-L_2	L_0-U_1	U_1-U_2	L_2-L_3	U_4-U_5	L_6-L_7 U_7-U_8	L_6-L_7 U_7-U_8	L_7-L_9	U_8U_9
Influence line........	1-6-30	1-6-30	1-10-30	1-10-30	1-16-30	1-25-30	1-15-18-30	1-15-18-30	1-15-18-30
Area................	126.47	126.47	108.13	108.13	71.47	16.47	20.2	20.2	40.4
Factor..............	36.67÷45	36.67÷28.5	73.33÷52.8	73.33÷55	146.67÷55	256.67÷55	256.67÷55	293.33÷55	330÷55
Equivalent Un. live load	2900	2900	2770	2770	2780	2750	2720	2730	2640
Live load stress......	+298	−472	−414	+400	−530	±211	±256	−294	+640
Impact stress........	+142	−225	−197	+190	−252	±100	±122	−140	+200

Web Members

Area for web members equals vertical component of shear in panel or vertical component of stress in member if chords are parallel (due to unit load per linear foot).

Factor reduces shear (see *area*) to stress in member (same loading). L.L. *Stress* equals product of area times factor times equivalent uniform L.L.

	U_1-L_2	U_2-L_2		L_2U_3	U_2L_3	L_3U_4	U_3L_4	U_4L_4	L_4U_5	U_4L_5	U_5L_5	L_5U_6	U_6L_6	L_6U_7	U_7L_7
Influence line........	38-10-30	1-7-36	36-13-30	1-8-37	37-13-30	1-11-35	35-16-30	1-11-35	1-14-34	34-19-30	1-14-34	1-17-33	1-17-33	1-20-32	1-20-32
Area................	95.5	36.6	66.1	9.4	68.5	26.6	44.6	26.6	47.0	26.0	47.0	69.3	69.3	98.0	98.0
Factor..............	$(^{58}/_{45})\times(^{3.5}/_{5.5})$ =.812	$^{3.5}/_{5.5}$ =.638	.638	$^{66}/_{55}$= 1.2	1.2	1.2	1.2	1.0	1.2	1.2	1.0	1.2	1.0	1.2	1.0
Equivalent Un. L.L...	3000	3470	3100	3550	3000	3380	3120	3380	3230	3240	3230	3100	3100	3000	3000
Live load stress......	+232	+81	−130	+40	+248	+108	+167	−90	+182	+102	−152	+258	−215	+353	−294
Impact stress........	+120	+62	−73	+31	+137	+76	+100	−63	+117	+66	−97	+152	−127	+192	−160
Equ. Un. L.L. = 2500 L.L. on other span.	....	+33	....	+60	....	+60	....	−50	+60	....	−50	+60	−50	+60	−50
L.L. stress due to L.L. on both spans.	....	+114	....	+100	....	+168	....	−140	+242	....	−202	+318	−265	+413	−344
Impact stress for above case.........	....	+49	....	+42	....	+67	....	−56	+90	....	−75	+113	−94	+140	−117

2–30. All influence lines are shown in Fig. 113 and areas may be scaled, but it is more accurate to calculate them. If the area under the influence line for the reaction is accurately calculated most other areas may be easily found by additions to or deductions from this area.

The method of influence lines and equivalent uniform live loads is to be recommended for the expeditious solution of any problem of this character.

CHAPTER V

THE RIGID FRAME

85. Preliminary.—A frame with rigid joints is a structure in which the member intersections are so constructed that the original angle between the members is maintained under any loading. Figs. 111*a* and 111*b* illustrate the difference between the ordinary hinge joint (for a steel frame) and the solid joint.

The very important practical question arises whether or not a truly rigid joint is realizable in practice. This point is discussed briefly in Chapter VIII. It is noted there that results of tests indicate that for a steel-frame joint with a heavy gusset plate and ample riveting, the assumption of complete rigidity is justifiable. For monolithic reinforced concrete construction the assumption is usually made without question.

By far the most important case of the rigid joint frame is to be found in the column and girder combination in building construction. The rapidly expanding use of monolithic reinforced concrete construction, however, gives an increasing number and variety of such problems—culverts, tunnel linings, A-frames for viaduct towers and many similar cases. It has been the custom in the past and is still so to a great extent, to analyze a building frame as independent beams and columns without due regard to its essentially monolithic character. It is gradually becoming recognized, however, that such a method is inadequate and often leads to serious errors, and that such a structure can only be safely and economically designed by treating it as a multiple rigid frame.

We have already, in Chapters II and III, solved a number of frame problems to illustrate the general method of attack on statically indeterminate structures, and the theory there developed will suffice for the solution of all rigid frame problems treated in this book. The present chapter will be largely devoted to the study of a number of more or less simple numerical problems in frame analysis, to certain cases of especial importance in building designs, and to a few special problems of a rather more complex character.

The slope-deflection method will prove by far the most effective

analytical instrument in the investigation of these problems, and will be used almost exclusively in this chapter. Some space will be devoted to restatement and amplification of the method.

86. Review and Restatement of Slope-Deflection Equations.—*Special Forms for the Case Where no Linear Displacement of the Joints Occurs.* We recall briefly the following salient points in the slope-deflection theory:

(1) The moments are expressed (separately for each member, regardless of the magnitude of the framework) as functions of the fixed beam end moments and certain elastic distortions—the tangential rotations, Θ, at the member-ends, and the rotation of the entire

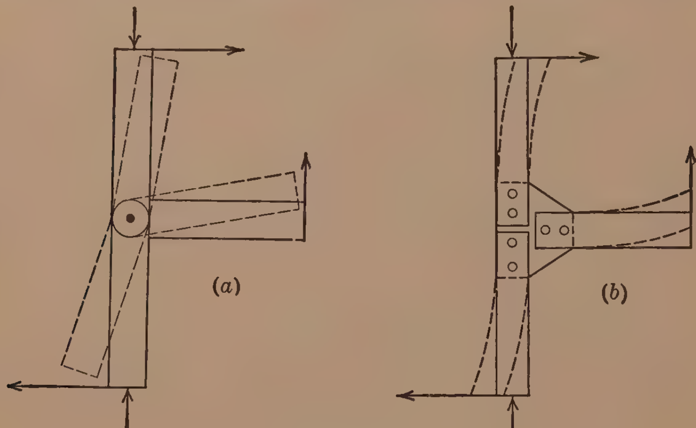


FIG. 111

member, R. The latter is equal to the relative linear displacement of the member-ends (normal to the axis) divided by the length ($D \div L$). The relation takes the general form

$$M_{mn} = M_{Fmn} + 2E\frac{I_{mn}}{L_{mn}}[3R_{mn} - 2\Theta_m - \Theta_n]. \quad . \quad . \quad (42)$$

(See Chapter III, page 149.)

(2) For a very large group of problems it will be found that the number of unknown Θ and R values is smaller (in many cases *much* smaller) than the unknown (statically undetermined) moments; therefore by means of such equations as (42) we reduce proportionately the number of unknowns to be obtained by a solution of simultaneous equations. *Herein lies the great advantage of the slope-deflection method.*

(3) When each unknown moment has been expressed in terms of R and Θ by means of such equations as (42), the relations between the

latter which are needed to effect a solution are obtained from *statical considerations*. The equations expressing these relations have been noted in Chapter III. They are, of course, identical in the fact that each expresses a necessary requirement for static equilibrium. In other respects they (for most cases) fall conveniently into two different groups: (a) one expressing the fact that the moments (at the member-ends) acting about any joint must sum up to zero, and (b) the other expressing the fact that in case a shear is transmitted from one level to another through a group of members, the summation of the end moments for each member divided by its length, must equal the shear transmitted. (If any member should have transverse loads acting at intermediate points, this effect must be provided for by an obvious modification.)

In a great many frame problems the loading is so applied that the relative linear displacements of the joints are negligible. For such cases R vanishes and, since the unknown Θ's cannot exceed the number of joints, equations of type (a) suffice for the complete solution. If translation occurs, we must have, in addition to the above set of equations, as many more as there are independent values of the relative end deflection, D. It will be convenient to designate type (a) as the "joint" equation, and type (b) as the "bent" equation. In most cases of practical importance the value of D is constant over the full width of a frame-bent for all members of a given story or panel, and hence in, say, a building frame we shall have one "bent" equation for each story of the bent, or for the case of an open-webbed girder, one equation for each panel.

The more general case where translation as well as rotation of the joints occurs will be treated further in Article 96. We shall here discuss the simpler case where the joints are subjected to rotation only, which case, as we have noted, governs a great variety of practical problems. We shall first note a convenient form for the expression for Θ—the simplified joint equation.

If we take a joint A (see Fig. 112) with members AB, AC, . . . AN framing into it, we must have

$$\Sigma M_{(\text{at } A)} = M_{AB} + M_{AC} + \ldots M_{AN} = 0,$$

and since

$$M_{AB} = M_{FAB} - 2EK_{AB}[2\Theta_A + \Theta_B], \quad \left(K = \frac{I}{L}\right),$$

and similarly for other moments, we must have

$$0 = \sum_{AN}^{AB} M_F - 2E\left[2\Theta_A \cdot \sum_{AN}^{AB} K + \Sigma(\Theta)_{N}^{B} \cdot (K)_{AN}^{AB}\right],$$

whence

$$E\Theta_A = \frac{\frac{1}{2}\sum\limits_{AN}^{AB} M_F - \sum\limits_{N}^{B}(\Theta)\cdot\underset{AN}{\overset{AB}{(K)}}}{2\sum\limits_{AN}^{AB} K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Since it is more convenient to deal with the $E\Theta$ product than with Θ itself, we shall in the remainder of this chapter use the symbol Θ to represent $E \times \Theta$ and $R = ER$. With this modification the slope-deflection equation will read

$$M_{mn} = M_{Fmn} + 2K_{mn}[3R_{mn} - 2\Theta_m - \Theta_n]. \quad . \quad . \quad . \quad (42a)$$

It may be well to note again the sign convention.

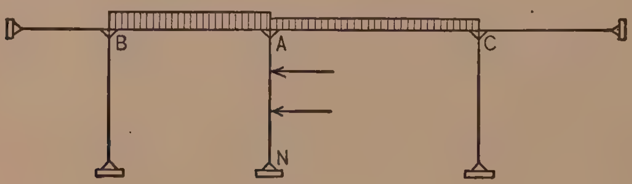


FIG. 112

(*a*) M_F is treated as positive whenever it tends to rotate the joint at which it acts in a clockwise direction.

(*b*) Θ, the angular displacement of any joint, is considered positive when clockwise.

(*c*) The final end moments, since they are expressed in terms of M_F and Θ, must correspond to the above sign convention; i.e., if any end moment works out positive, it is to be interpreted as tending to give the corresponding joint a clockwise rotation.

It will be clear upon a little reflection that the *actual* $K\left(= \frac{I}{L}\right)$ values are unnecessary; if *relatively* correct values are used the final moments will be correct. That is to say, if

$$K_{AB} = 22.5 \qquad K_{AC} = 45 \qquad K_{AN} = 60$$

it will suffice to use

$$K_{AB} = 1 \qquad K_{BC} = 2 \qquad K_{AN} = \tfrac{8}{3} = 2.66.$$

87. Example.—In Fig. 113 we have a frame of four members of varying lengths and stiffnesses. All joints except A are assumed com-

pletely fixed; A is free to rotate, but is assumed to have no linear displacement. Since all Θ values except Θ_A vanish, equation (43) becomes

$$\Theta_A{}^* = \frac{\frac{1}{2}M_{FAB}}{2\sum\limits_{AE}^{AB}(K)} = \frac{PL_{AB}(k_B{}^2 - k_B{}^3)}{4[K_{AB} + K_{AC} + K_{AD} + K_{AE}]}.$$

If we assume numerical data as follows:

$$P = 10{,}000\#; \qquad L = 12'\text{-}0'' \qquad k = \tfrac{3}{4},$$

and

$$K_{AB} = 1; \qquad K_{AC} = 3; \qquad K_{AD} = 2; \qquad K_{AE} = 4.$$

Then

$$M_{FAB} = 10{,}000 \times 144(.75^2 - .75^3) = 202{,}000''\#$$

$$\Theta_A = \frac{202{,}000''\#}{4(1 + 3 + 2 + 4)} = +\,5050.$$

$$\left.\begin{array}{ll} M_{AB} = +\,202{,}000''\# - 2(2 \times 5050) = & +\,181{,}900 \\ M_{AC} = \quad -\,2 \times 3(2 \times 5050) = & -60{,}700 \\ M_{AD} = \quad -\,2 \times 2(2 \times 5050) = & -40{,}400 \\ M_{AE} = \quad -\,2 \times 4(2 \times 5050) = & -80{,}800 \end{array}\right\} = -\,181{,}900 \text{ check}$$

The fact that the clockwise rotating tendency of M_{AB} is just equal to the sum of the counterclockwise rotating tendencies of the other three moments, checks up the condition for equilibrium of moments at A, upon which the solution is based.

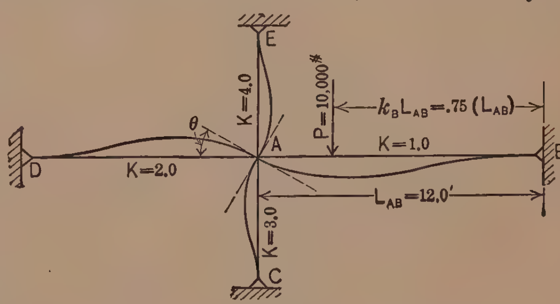


Fig. 113

We may then write the values of moments at the fixed ends B, C, D, and E

$$M_{BA} = -\,10{,}000 \times 144(.25^2 - .25^3) - 2(5050) =$$
$$-\,1{,}440{,}000(.0469) - 10{,}100 = -\,77{,}600''\#$$
$$M_{CA} = -\,2 \times 3(5050) = -\,30{,}300''\#$$
$$M_{DA} = -\,2 \times 2(5050) = -\,20{,}200''\#$$
$$M_{EA} = -\,2 \times 4(5050) = -\,40{,}400''\#.$$

* Actually $E\Theta$, but see page 204.

It is seen from the foregoing calculations that in such a problem, although there are *eight* different moments to be found, their values may be written out with very little calculation, and no solving of simultaneous equations is necessary.*

88. Shear.—It is frequently of importance to construct the shear diagram as well as the moment diagram. We shall for this purpose

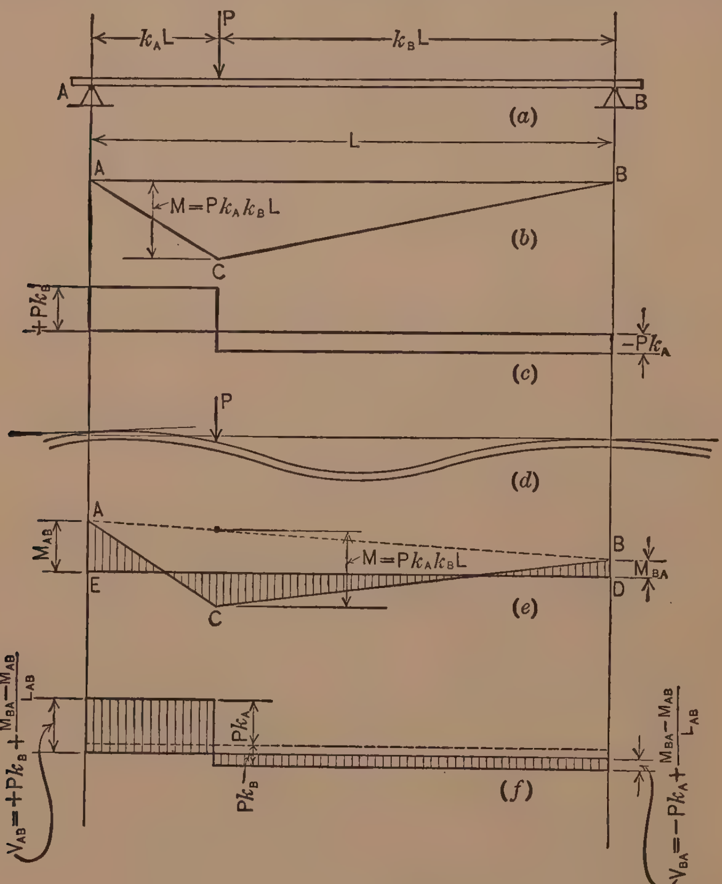


FIG. 114

use the ordinary sign convention rather than the special convention of the slope-deflection equations. The subject was discussed briefly in Chapter IV, pp. 164–5. We may illustrate the method by reference to Fig. 114. Fig. 114*a* shows *AB* as a simple beam, and 114*b* and 114*c* show corresponding moment and shear diagrams. Fig. 114*d* illustrates

* Compare with similar problem in Chapter III, pages 143–4.

the case where continuity exists at A and B and (e) and (f) show the corresponding moment and shear diagrams. It is clear that the final shear diagram (f) is equal to the simple beam shear diagram (c) modified by the shear induced by the end moments. Calling this latter ΔV, and calling all moments positive or negative according as they correspond to compression or tension on the top fiber, and shears positive or negative according as the resultant force acts upward or downward to the left of the section, we shall have

$$V_{AB} = V'_{AB} + \Delta V_{AB}, \qquad \text{if } V'_{AB} = \text{simple beam shear}$$

and

$$\Delta V_{AB} = \frac{M_{BA} - M_{AB}}{L_{AB}}.$$

Likewise

$$V_{BA} = V'_{BA} + \Delta V_{BA} = V'_{BA} + \frac{M_{AB} - M_{BA}}{L_{AB}}$$

This method of writing the shear equation is quite general; we have used a horizontal member and corresponding terms ("upward," "top," etc.) for illustration, but it is clear that we may, for purposes of analysis, treat any member as horizontal without loss of generality. The point to be noted particularly is that if the moments have been obtained by the slope-deflection method, their signs must be modified to conform to the ordinary rules before they are introduced into the preceding equations.

For member AB of the preceding problem we have

$$V'_{AB} = +7500 \qquad M_{AB} = -151{,}830$$

$$V'_{BA} = -2500 \qquad M_{BA} = -\ 64{,}730$$

$$V_{AB} = +(+7500) + \frac{+(-64{,}730) - (-151{,}830)}{120} = +8225$$

$$V_{BA} = +(-2500) + \frac{+(-64{,}730) - (-151{,}830)}{120} = -1775$$

Check:

$$V_{AB} - V_{BA} = P \qquad 8225 - (-1775) = 10{,}000.$$

89. The Frame Building Bent under Vertical Loads.—*Example* 1. We may get an approximate solution to this problem, showing in a general way the effect of an isolated uniform loading on an interior beam in producing secondary bending in the column and beam ends framing into this beam, by assuming all joints as fixed except those at the ends

of the member considered, which are assumed free to rotate (see Fig. 115). We have, from symmetry, $\Theta_B = -\Theta_A$ and

$$\Theta_A = \frac{\frac{1}{2}\sum_{AN}^{AB} M_{FAB} - \sum_{AN}^{AB}(K)\cdot(\Theta)_{N}^{B}}{2\sum_{AN}^{AB}(K)}. \quad . \quad . \quad . \quad . \qquad (44)$$

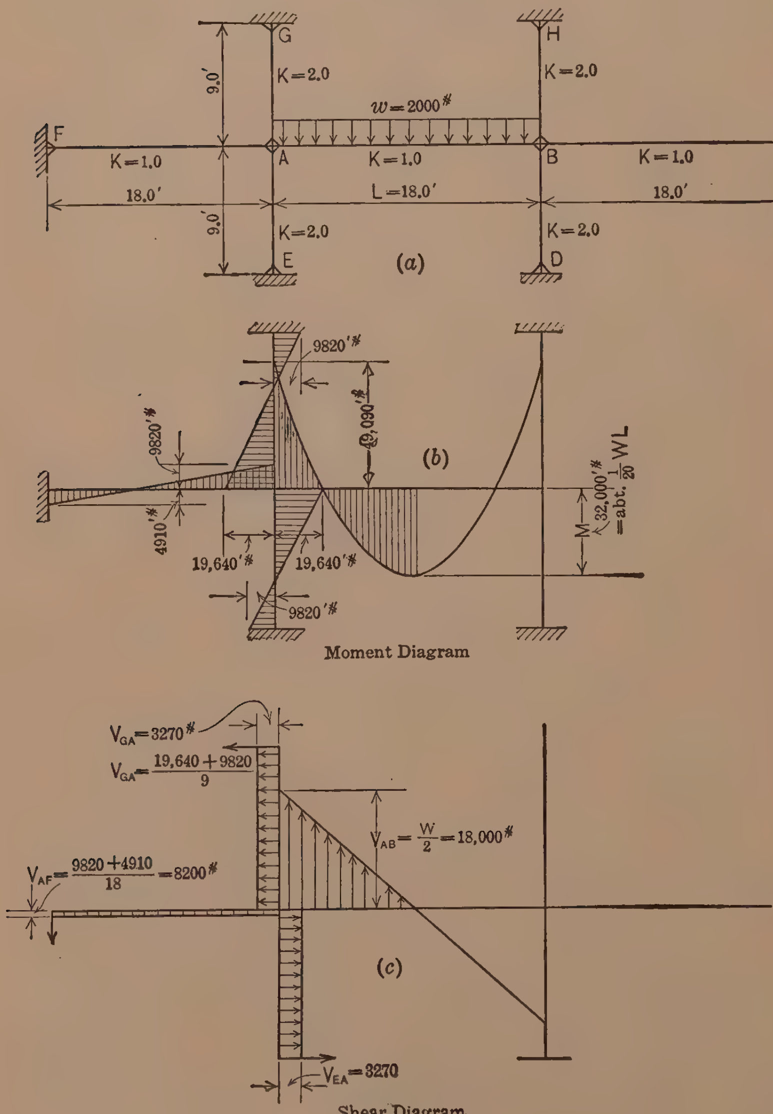


FIG. 115

Since

$$M_F = \tfrac{1}{12} \times 2000 \times 18^2 = 54{,}000'\#$$

$$\theta_A = \frac{\frac{1}{2} \times 54{,}000 - 1.0 \cdot \theta_B}{2(1 + 2 + 1 + 2)} = \frac{27{,}000 - (-\theta_A)}{12}.$$

Whence

$$\theta_A = +2455 = -\theta_B.$$

The end moments then are

$$\left.\begin{aligned} M_{AB} &= 54{,}000 - 2 \times 2455 &&= +49{,}090 \\ M_{AE} &= -2 \times 2(2 \times 2455) &&= -19{,}640 \\ M_{AF} &= -2 \times (2 \times 2455) &&= -9{,}820 \\ M_{AG} &= -2 \times 2 \times (2 \times 2455) &&= -19{,}640 \end{aligned}\right\} \Sigma M_A = -10$$

$$\begin{aligned} M_{EA} &= -2 \times 2(2455) &&= -9{,}820 \\ M_{FA} &= -2 \times (2455) &&= -4{,}910 \\ M_{GA} &= -2 \times 2 \times 2455 &&= -9{,}820 \end{aligned}$$

The extraordinary simplicity of the slope-deflection solution for such cases is apparent from this example.

There is of course some error involved in the assumption of complete restraint at all joints except A and B. For an approximate solution, however, the assumption gives satisfactory results.

90. *Example* 2.—Figs. 115*d* and 115*e* illustrate a problem analytically very similar to the preceding. A heavy side wall and heavy footings are assumed to practically fix the outer ends of the beams, F and F', and the column bases.

The problem is very simply solved by slope-deflections and also has a fairly simple solution by moment areas which will be given also for sake of comparison.

(*a*) *Solution by Slope-Deflection Method.*

Since from symmetry

$$\theta_B = -\theta_B' = \theta$$

we have

$$M_E = -2K_1(2\theta)$$

$$M_C = -2K_2(2\theta)$$

$$M_B = -2K(\theta) + \frac{wL^2}{12}$$

Also

$$M_E + M_C + M_B = 0,$$

whence

$$\theta = \frac{\dfrac{wL^2}{12}}{2[K_1 + 2K_2 + K]}$$

$$M_B = \frac{wL^2}{12}\left[\frac{2\dfrac{I_1}{L_1} + 2\dfrac{I_2}{L_2}}{2\dfrac{I_1}{L_1} + 2\dfrac{I_2}{L_2} + \dfrac{I}{L}}\right]$$

$$M_C = -\frac{wL^2}{12}\cdot\frac{2\dfrac{I_1}{L_1}}{2\dfrac{I_1}{L_1} + 2\dfrac{I_2}{L_2} + \dfrac{I}{L}}$$

$$M_E = -\frac{wL^2}{12}\cdot\frac{2\dfrac{I_2}{L_2}}{2\dfrac{I_1}{L_1} + 2\dfrac{I_2}{L_2} + \dfrac{I}{L}}.$$

(*b*) *Solution by Moment Areas.*

As sketched in Fig. 115*d* there are six unknown bending moments, M_A, M_B, M_C, M_D, M_E, and M_F. Between these six unknowns, we can only set up two equations from the conditions of statics. These are

$$M_B = M_E + M_C. \quad \ldots \ldots \ldots \quad (a)$$

$$M_A + M_B = \frac{wL^2}{8}. \quad \ldots \ldots \ldots \quad (b)$$

The remaining four equations can be written with the aid of the moment area principle. The distribution of the $\frac{M}{EI}$ values is indicated by the curves of the diagram Fig. 115*e*.

The condition of rigidity of the joint at B requires that the two tangents at this point shall always be perpendicular to each other. From this condition we get two independent equations:

$$\frac{Q}{L} = \frac{Q_2}{L_2} \quad \text{and} \quad \frac{Q_1}{L_1} = \frac{Q_2}{L_2}.$$

Here Q, Q_1, and Q_2 are the deflections of the points B', D and F. The deflection Q is measured by the statical moment of the area

1–5–3–4–2, Fig. 115*e* about B', or the moment of the area 1–5–3 minus the moment of the area 1–2–4–3.

Therefore

$$Q = \frac{\frac{wL_2}{8} \times \frac{2}{3}L \times \frac{1}{2}L - M_B \times \frac{1}{2}L \times L}{EI}.$$

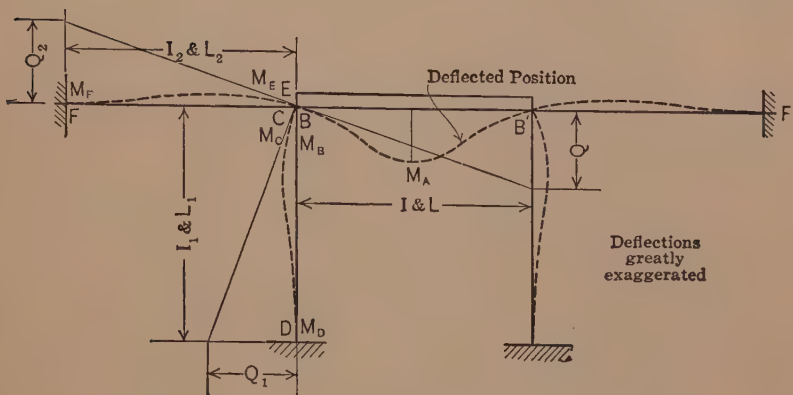


FIG. 115*d*

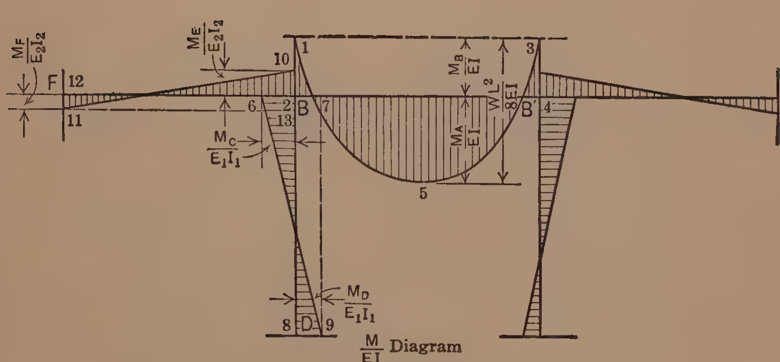


FIG. 115*e*

The deflection Q_2 is measured by the statical moment of the area 12–2–10–11 about the point F, equal to the moment of the area 11–13–10 minus the moment of the area 12–11–13–2.

Therefore

$$Q_2 = \frac{(M_E + M_F)\frac{L_2}{2} \times \frac{2}{3}L_2 - M_F \times L_2 \times \frac{L_2}{2}}{E_2I_2}.$$

And knowing that $\frac{Q}{L}$ equals $\frac{Q_2}{L_2}$ we get

$$\frac{\frac{wL^3}{3 \times 8} - \frac{M_B L}{2}}{EI} = \frac{\frac{(M_E + M_F)L_2}{3} - \frac{M_F L_2}{2}}{E_2 I_2},$$

whence, if E is constant,

$$\frac{wL^2}{4} - 3M_B = K'_1(2M_E - M_F), \quad \ldots \ldots \quad (c)$$

if

$$\frac{L_2 I}{L I_2} = K'_1.$$

Also since $\frac{Q_1}{L_1}$ equals $\frac{Q_2}{L_2}$ we get

$$\frac{\frac{(M_C + M_D)L_1}{3} - \frac{M_D L_1}{2}}{E_1 I_1} = \frac{\frac{(M_E + M_F)L_2}{3} - \frac{M_F L_2}{2}}{E_2 I_2},$$

whence, if E is constant

$$2M_C - M_D = K'_2(2M_E - M_F), \quad \ldots \ldots \quad (d)$$

if

$$K'_2 = \frac{L_2 I_1}{L_1 I_2}.$$

The deflection of the point B from the tangent to the elastic curve at D is zero since the point B remains unchanged and the tangent at D is fixed by the conditions of the problem. Therefore, the statical moment of the area 8–2–6–9 about the point B is zero, or, what is the same, the moment of the area 6–9–7 about the point B minus the moment of the area 2–7–9–8 about the point B is equal to zero. From this we get

$$\frac{(M_C + M_D)\left(\frac{L_1}{2}\right)\left(\frac{L_1}{3}\right) - M_D L_1\left(\frac{L_1}{2}\right)}{E_1 I_1} = 0,$$

or

$$M_C = 2M_D. \quad \ldots \ldots \quad (e)$$

The same being true with regard to the tangent at the point F we may immediately write

$$M_E = 2M_F. \quad \ldots \ldots \quad (f)$$

Substituting (e) and (f) in (d) we get,

$$M_C = K'_2 M_E. \quad \ldots \ldots \quad (a')$$

Substituting (a') in (a) we get

$$M_B = (1 + K'_2)M_E. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b')$$

Substituting (b') and (f) in (c) we get,

$$\frac{wL^2}{4} - 3(1 + K'_2)M_E = \frac{3K'_1}{2}M_E.$$

Therefore

$$M_E = \frac{wL^2}{12}\left(\frac{2}{K'_1 + 2K'_2 + 2}\right)$$

$$= \frac{wL^2}{12}\left(\frac{2LL_1I_2}{L_1L_2I + 2LL_2I_1 + 2LL_1I_2}\right)$$

$$= C\frac{wL^2}{12},$$

where

$$C = \frac{2LL_1I_2}{L_1L_2I + 2LL_2I_1 + 2LL_1I_2} = \frac{2}{K'_1 + 2K'_2 + 2}.$$

From (a')

$$M_C = CK'_2\left(\frac{wL^2}{12}\right) = \frac{wL^2}{12}\left(\frac{2LL_2I_1}{LL_2I + 2LL_2I_1 + 2LL_1I_2}\right).$$

From (f)

$$M_F = \frac{C}{2}\left(\frac{wL^2}{12}\right).$$

From (e)

$$M_D = \frac{CK'_2}{2}\cdot\frac{wL^2}{12} = \frac{wL^2}{12}\left(\frac{LL_2I_1}{L_1L_2I + 2LL_2I_1 + 2LL_1I_2}\right)$$

$$= C'\frac{wL^2}{12},$$

if

$$C' = \frac{LL_2I_1}{L_1L_2I + 2LL_2I_1 + 2LL_1I_2}.$$

From (a)

$$M_B = C(1 + K'_2)\frac{wL^2}{12}$$

$$= \frac{wL^2}{12}\left(\frac{2LL_1I_2 + 2LL_2I_1}{L_1L_2I + 2LL_2I_1 + 2LL_1I_2}\right).$$

From (b)

$$M_A = \frac{wL^2}{8} - M_B.$$

It will be seen that the preceding moment values readily reduce to the same form as those of solution (a).

It is an easy matter to test some of these values for the limiting

cases. Take for instance the value for M_B. There are four conditions of the members BD and BF which would produce a condition of fixed ends for the central span BB'. If I_1 or I_2 becomes very large in comparison with the remaining values of I, or if L_1 or L_2 becomes very short in comparison with the remaining values of L, we would get a condition approaching fixed ends for the central span. In all four of these cases it is seen that the value of M_B approaches $\frac{wL^2}{12}$ which is the value of the end moment for a fixed span with a uniform load. When the values of I and L approach the other extreme we have a condition of a freely supported simple span, where M_B approaches zero, as a substitution will show.

When I_1 equals zero, I equals I_2, and L equals L_2, we have the con-

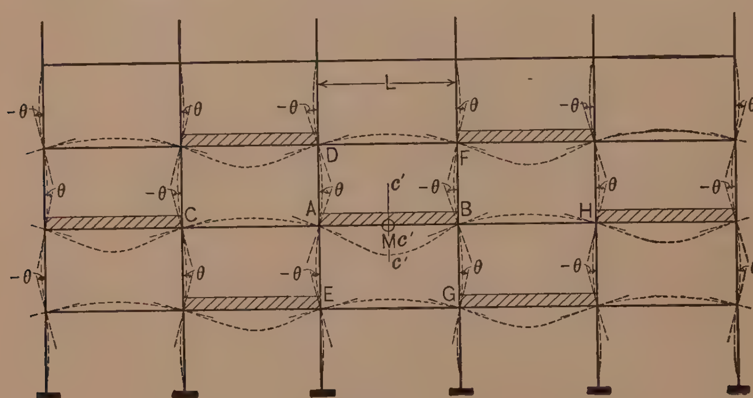


Fig. 116

dition of a beam of uniform section with three equal spans, fixed at the two end supports and supported freely on the two intermediate supports. Here again the values of M_B and M_E check themselves by becoming $\frac{wL^2}{18}$ while M_F becomes $\frac{wL^2}{36}$.

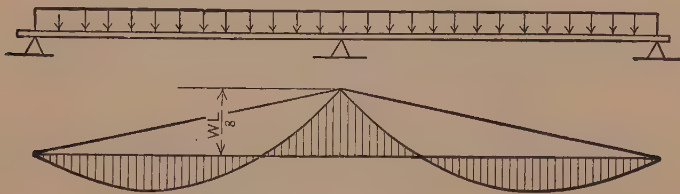
91. Maximum Moments in Interior Columns and Beams of Monolithic Frames.—(*a*) To obtain the maximum positive moments, or moments at the span center causing maximum tension on the bottom beam fibers, we find the case treated in Fig. 115 giving results too low. It will appear that when the loading is applied as indicated in Fig. 116 we have the condition most conducive to the maximum positive moment at $c - c$. In this case every Θ value will be of such sign that each one will contribute its increment to the moment considered.

Let us take the case of uniform live loading on the span in question

and the same live loadings on alternate spans of the same floor and adjacent spans of the floors above and below (see Fig. 116).

The expression for Θ may be written as in the preceding case, knowing that all Θ values will be of approximately the same numerical value, although differing in sign.

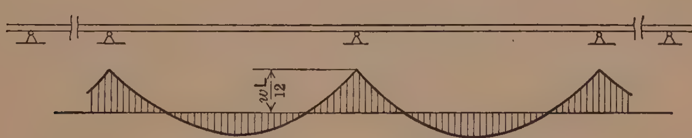
(*b*) For maximum negative moments in the continuous beams of a



$$2M_B(2L) = -\frac{WL}{4} - \frac{WL}{4}$$
$$M_B = -\frac{WL}{8}$$

FIG. 117

framed bent of a building it is evident that full live load on two adjacent spans *only* will give these maximums in beams at the joint which is common to these spans. The actual distribution is between that for the condition indicated in Fig. 117 and that of Fig. 118. The value in general would be much closer to $\frac{1}{12}WL$ than to $\frac{1}{8}WL$. Only in such a case as that of the two-span building bent, Fig. 119, where the exterior ends merely rest on bearing walls, instead of being monolithic with the



Assuming $M_A = M_B = M_D$

$$ML + 2M(2L) + ML = -\frac{wL}{4} - \frac{wL}{4} \quad \text{or} \quad M = -\frac{wL}{12}$$

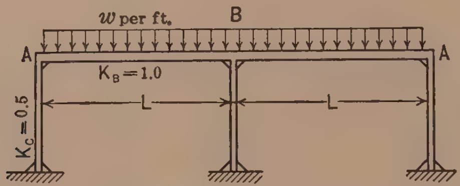
FIG. 118

columns as shown in figure, would the value be $\frac{1}{8}WL$. This case sometimes arises, but it is not the usual case.

For skeleton steel or monolithic construction the condition indicated in Fig. 119 would give a value of $\frac{1}{9}WL$—a value only 11 per cent smaller than the preceding. For more than two continuous spans in monolithic construction, cases where the stiffness of columns increases relatively to that of beams, or where live loads become relatively smaller in comparison

with dead loads, we may expect values of the maximum negative moment which are much closer to $\frac{1}{12}WL$.

92. Moments in Exterior Columns and Beams of Monolithic Frames. —A condition which gives rise to large bending moments in exterior



$$\theta_A = \frac{+\frac{1}{2}(\frac{1}{12}wL^2)}{2(1.5)} = +\frac{1}{72}wL^2$$

$$M_{BA} = -\frac{1}{12}wL^2 - 2(1.0)(\frac{1}{72}wL^2) = -\frac{1}{9}wL^2$$

FIG. 119

columns is indicated in Fig. 120. The particular moments referred to are M_{AC} and M_{AG}.

One of the assumptions which gives a short solution for these exterior

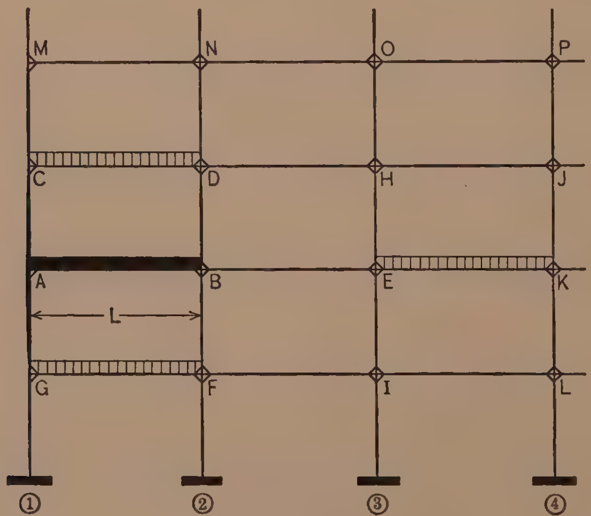


FIG. 120

column moments is illustrated in Fig. 121 where the joints C, D, E, F, and G are assumed to be fixed against rotation.

The unknown moment values will depend in large degree upon

the relative stiffness $\left(K = \frac{I}{L}\right)$ values of the columns and beams. We shall here assume a case where the columns are relatively quite stiff—$K_{\text{Column}} = 2$ as compared with $K_{\text{Beam}} = 1$.

In this problem Θ_A and Θ_B are the only values necessary to the

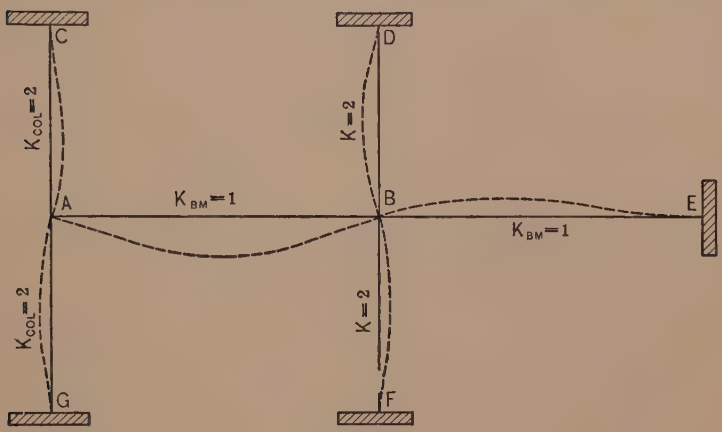


FIG. 121

writing of all the end moments, and to obtain these values we have the two conditions that

$$\Sigma M_A = 0 = \Sigma M_B.$$

Let W = total load on AB,

then

$$M_{AB} = M_{FAB} - 2K_{Bm}(2\Theta_A + \Theta_B) = +\frac{WL}{12} - 2(2\Theta_A + \Theta_B)$$

$$M_{AC} = -2K_{Col}(2\Theta_A) = -4(2\Theta_A) = M_{AG}$$

$$\Sigma M_A = 0 = \frac{WL}{12} - 20\Theta_A - 2\Theta_B,$$

also

$$M_{BA} = -M_{FBA} - 2K_{Bm}(2\Theta_B + \Theta_A) = -\frac{WL}{12} - 2(2\Theta_B + \Theta_A)$$

$$M_{BD} = -2K(_{Col}2\Theta_B) = -4(2\Theta_B)$$

$$M_{BE} = -2K_{Bm}(2\Theta_B) = -2(2\Theta_B)$$

$$M_{BF} = -2K_{Col}(2\Theta_B) = -4(2\Theta_B)$$

$$\Sigma M_B = 0 = -\frac{WL}{12} - 24\Theta_B - 2\Theta_A.$$

Solving simultaneously

$$20\Theta_A + 2\Theta_B = +\frac{WL}{12},$$

$$2\Theta_A + 24\Theta_B = -\frac{WL}{12},$$

$$\Theta_A + .10\Theta_B = \frac{WL}{240},$$

$$\Theta_A + 12.00\Theta_B = -\frac{WL}{24},$$

$$11.9\Theta_B = -\frac{11WL}{240},$$

$$\Theta_B = -.00385WL,$$

$$\Theta_A = +.00416WL - .10(-.00385WL)$$

$$= +.00454WL.$$

Using Θ_A we find the moment coefficient of M_{AC} and M_{AG} as follows:

$$M_{AG} = M_{AC} = -8\Theta_A = .0363WL = -\frac{1}{27.5}WL.$$

As a check on the coefficients we know that for equilibrium at the joint A,

$$M_{AG} + M_{AC} + M_{AB} = 0, \quad \text{or} \quad M_{AB} = +\frac{1}{13.75}WL,$$

$$M_{AB} = +\frac{WL}{12} - 2[2(+.00454)WL + (-.00385WL)]$$

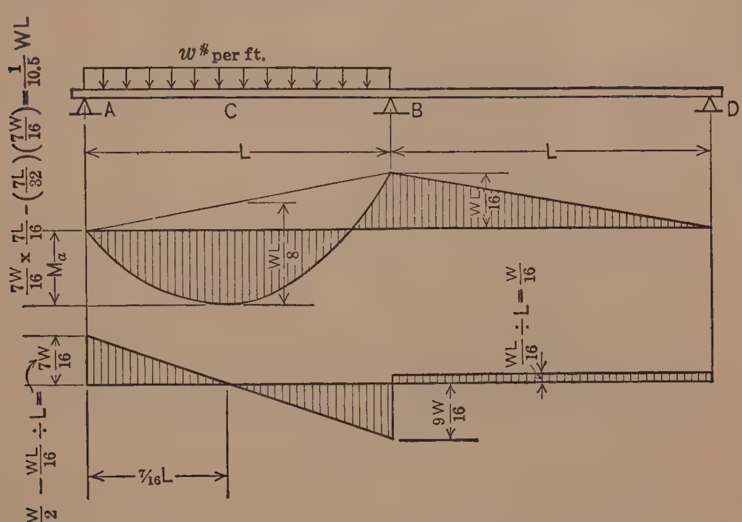
$$= .0729WL = \frac{1}{13.72}WL.$$

This gives a convenient check on the numerical work.

Values of maximum positive moments will approach $\frac{WL}{10.5}$ where the stiffness of the columns vanishes and the live load is large compared to the dead load. We then have the case of a two-span continuous beam resting on exterior supports with only one span loaded. (See Fig. 122.)

The conditions illustrated in Fig. 123 more nearly approach the actual conditions in a multiple-story bent, and would tend to increase the exterior column moments to a value somewhat greater than $\frac{1}{27.5}$. We shall now analyze this more exact case for exterior column moments

and also investigate the effect of reducing the column stiffness to one-half (instead of twice) the beam stiffness.



By the 3-moment eq., . . . $2M_B(L + L) = -\frac{WL}{4}$, or $M_B = -\frac{WL}{16}$.

FIG. 122

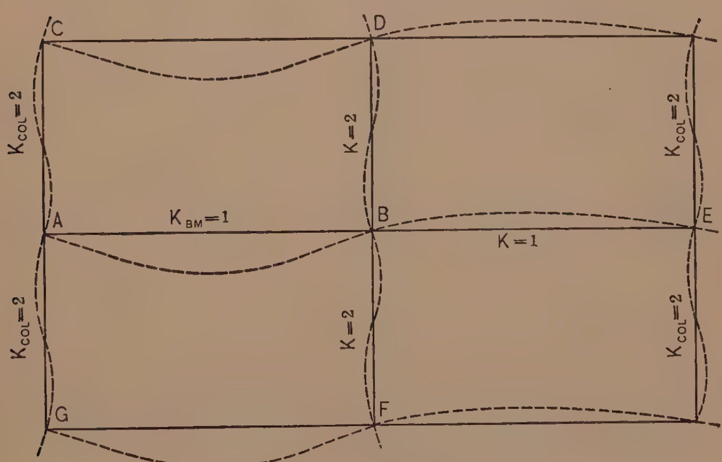


FIG. 123

For this problem we may assume

$$\theta_A = \theta_C = \theta_G,$$

and

$$\theta_B = \theta_F = \theta_D = -\theta_E.$$

Then

$$M_{AB} = + M_{FAB} - 2K_{Bm}(2\Theta_A + \Theta_B) = \frac{WL}{12} - 2(2\Theta_A + \Theta_B),$$

$$M_{AC} = - 2K_{Col}(2\Theta_A + \Theta_C) = - 1(3\Theta_A),$$

$$M_{AG} = - 2K_{Col}(2\Theta_A + \Theta_G) = - 1(3\Theta_A),$$

$$\Sigma M_A = 0 = + \frac{WL}{12} - 10\Theta_A - 2\Theta_B,$$

and

$$M_{BA} = - M_{FBA} - 2K_{Bm}(2\Theta_B + \Theta_A) = - \frac{WL}{12} - 2(2\Theta_B + \Theta_A),$$

$$M_{BD} = - 2K_{Col}(2\Theta_B + \Theta_D) = - 1(3\Theta_B),$$

$$M_{BE} = - 2K_{Bm}(2\Theta_B + \Theta_E) = - 2(\Theta_B),$$

$$M_{BF} = - 2K_{Col}(2\Theta_B + \Theta_F) = - 1(3\Theta_B),$$

$$\Sigma M_B = 0 = - \frac{WL}{12} - 12\Theta_B - 2\Theta_A.$$

Solving simultaneously

$$10\Theta_A + 2\Theta_B = \frac{WL}{12},$$

$$2\Theta_A + 12\Theta_B = - \frac{WL}{12},$$

$$\Theta_A + .20\Theta_B = .00833WL,$$

$$\Theta_A + 5.8\Theta_B = - .0416WL,$$

$$5.8\Theta_B = - .0499WL,$$

$$\Theta_B = - .0086WL.$$

$$\Theta_A = .00833WL - .20(- .0086)WL = + .01WL,$$

whence

$$M_{AG} = - .03WL = \tfrac{1}{33}WL.$$

Basing the argument upon the two preceding examples, which cover a considerable range of stiffness conditions, we might generalize and say that for usual conditions we should expect an exterior column moment (at an intermediate floor) of about $\frac{1}{30}WL$ where W and L are the total load and span respectively, for the adjacent beam.

The negative end moment value (where the beam frames into the side column) can naturally be expected to be about twice that of the exterior column moment or about $\frac{WL}{13}$ to $\frac{WL}{17}$ (since in this case one beam

moment must hold the column moments developed both above and below the joint in equilibrium).

In the above as in the several preceding cases, more accurate results may be obtained by assuming a greater number of joints surrounding the one under investigation as free to rotate. Thus in Fig. 120, we might assume each of the nine joints A, B, . . . H, . . . I as free to turn. The procedure is in principle identical with that illustrated in the given problems, except of course that it is much more tedious and lengthy since a much larger set of simultaneous equations must be solved. In some cases the added accuracy may be of decided practical importance. Usually, however, the approximate solutions previously presented will suffice for the designer's needs.

The above conclusions also give an idea of what to expect in the exterior columns of flat-slab construction. Westergaard and Slater in their notable paper * on flat-slab analysis and tests conclude that the flat slab-panel framing into columns may be analyzed in the same fashion and about as accurately as the ordinary beam-column bent. This being so we have a right to expect a moment of something less than $\frac{1}{30}wL^3$ in the exterior columns of a flat-slab structure. As a matter of fact more attention has been paid by investigators to the problem of exterior columns in the flat slab-and-column bent than in the beam-column bent.

93. A Study of the Reinforced Concrete Frame for Design Moment Coefficients.—In ordinary buildings of reinforced concrete where the construction is monolithic and the joint is rigid, each isolated beam or column is a frame member. This continuity of construction makes economies in reinforcing steel possible and this steel is universally detailed so as to provide such continuity.

The design of beams and girders in ordinary structures (where all spans and loading are usually symmetrical) is generally simplified by certain assumptions. For example, in the average interior span one would first calculate the ordinary simple beam moment. A correction factor would then be applied to provide for the reducing effect of the continuity on this value. For full live loading of adjacent spans it would be assumed that the negative end moment of the interior beam would approach $\frac{2}{3}$ of the M_s (simple beam) value. When the live load is removed from adjacent spans only, then the positive center moments of the span would increase and would approach the same value of $\frac{2}{3}M_s$.

* "Moments and Stresses in Slabs," by H. M. Westergaard and W. A. Slater; Proceedings of American Concrete Institute, Vol. XVII, pp. 430, et seq.

Similarly in exterior spans a value of $\frac{4}{5}M_s$ might be approached either at the center section or at the interior end section.

Several problems will now be worked out to indicate the possible distribution of beam moments between center and end sections for various assumptions as to loading and stiffness. It will be interesting to compare these results with the recommendations of the "Joint Committee," 1921:

The following moments at critical sections of freely supported beams and slabs of equal spans carrying uniformly distributed loads will be used:

(*a*) Maximum positive moment in beams and slabs in one span,

$$M = \frac{wL^2}{8}. \qquad (12)$$

(*b*) Center of slabs and beams continuous for two spans only,

(1) Positive moment at the center,

$$M = \frac{wL^2}{10}. \qquad (13)$$

(2) Maximum negative moment,

$$M = \frac{wL^2}{8}. \qquad (14)$$

(*c*) Slabs and beams continuous for more than two spans,

(1) Center and supports of interior spans,

$$M = \frac{wL^2}{12}. \qquad (15)$$

(2) Center and interior support of end spans,

$$M = \frac{wL^2}{16}. \qquad (16)$$

(*d*) Negative moment at the supports of slab or beam built into brick or masonry walls in a manner that develops partial end restraint,

$$M = \text{not less than } \frac{wL^2}{16}. \qquad (17)$$

The following moments at the critical sections of beams or slabs of equal spans cast monolithic with columns or similar supports and carrying uniformly distributed loads shall be used:

(*a*) Supports of intermediate spans,

$$M = \frac{wL^2}{12}. \qquad (18)$$

(*b*) Center of intermediate spans,

$$M = \frac{wL^2}{16}. \qquad (19)$$

(*c*) Beams in which $\frac{I}{L}$ is less than twice the sum of the values of $\frac{I}{h}$ for the exterior columns above and below which are built into the beam.

(1) Center and first interior support,

$$M = \frac{wL^2}{10} \quad \text{(20)}$$

(2) Exterior supports,

$$M = \frac{wL^2}{12} \quad \text{(21)}$$

(*d*) Beams in which $\frac{I}{L}$ is equal to, or greater than, twice the sum of the values of $\frac{I}{h}$ for the exterior columns above and below which are built into the beam.

(1) Center of span and at first interior support of end span.

$$M = \frac{wL^2}{10} \quad \text{(22)}$$

(2) Exterior support,

$$M = \frac{wL^2}{16} \quad \text{(23)}$$

Continuous beams with unequal spans, whether freely supported or cast monolithic with columns, shall be analyzed to determine the actual moments under the given conditions of loading and restraint. Provision shall be made for negative moment occurring in short spans adjacent to longer spans when the latter only are loaded.

94. *Variation of Moment of Inertia.*—There is a particular difficulty encountered in reinforced concrete frame analysis in the matter of estimating the value of $\frac{I}{L}$. The moment of inertia of a reinforced concrete member subjected to direct stress and bending, even when the dimensions and reinforcing are constant, will exhibit marked discontinuous changes, due to the cracking of the concrete in tension where the bending stress is predominant, and any exact mathematical representation of the variation seems out of the question. Other factors difficult to estimate precisely, such as variations in the placing of the steel areas, add to the difficulty of exactly determining the I-value at any section. However, for the purpose of determining the moment distribution, there has been proposed * the empirical formula

$$I \text{ varies as } bd^{2\frac{1}{2}}$$

where b and d have their usual significance in reinforced concrete design.

* So far as the authors know this formula was first proposed and used by W. A. Slater and F. E. Richart in their analytical and experimental study of reinforced concrete frames for the U. S. Shipping Board.

This formula was derived from a fairly comprehensive set of experiments, and it fitted the results very satisfactorily. It may be shown that it requires a very considerable percentage variation of the relative $\frac{I}{L}$ values of the members of a frame to effect an appreciable percentage variation in the final moments, which is an additional justification for the use of an approximate empirical formula.

95. *Maximum Positive Moment Calculations—Interior Spans.*—In the average multi-story building we have a wide range in the ratio of column to beam stiffness . . . $\frac{K_{Col}}{K_{Bm}}$. We shall consider two possible points in this range, one where the column $\frac{I}{L}$ equals twice the beam $\frac{I}{L}$ and another where the column $\frac{I}{L}$ equals one-half of the beam $\frac{I}{L}$. In both cases let $W = 20{,}000\#$ and $L = 20'$, (Fig. 116).

In the first case let

$$K_{Bm} = 1,$$

then

$$K_{Col} = 2,$$

$$M_{AB} = \frac{WL}{12} - 2K_{Bm}\Theta,$$

$$M_{AC} = -2K_{Bm}\Theta,$$

$$M_{AD} = -2K_{Col}\Theta,$$

$$M_{AE} = -2K_{Col}\Theta.$$

In all the above moment expressions the Θ's at the opposite ends of the member have same numerical value and opposite signs,—therefore

$$2K(2\Theta_A + \Theta_B) \text{ simplifies to } 2K\Theta,$$

and since

$$M_{AB} + M_{AC} + M_{AD} + M_{AE} = 0,$$

we have

$$\frac{WL}{12} = \Theta(4K_{Bm} + 4K_{Col}),$$

and

$$\Theta = \frac{WL}{12(4K_{Bm} + 4K_{Col})} = \frac{400{,}000}{12(4 + 8)} = 2780.$$

Therefore

$$M_{AB} = \frac{400{,}000}{12} - 2(2780) = 33{,}330 - 5560 = 27{,}770\#'$$

and

$$M_{C'} = \frac{WL}{8} - M_{AB} = 50{,}000 - 27{,}770 = 22{,}230\text{#}'$$

$$= \frac{22.23}{50.00} \times \frac{WL}{8} = \frac{WL}{18} \text{ (about).}$$

When

$$K_{Col} = \tfrac{1}{2} \quad \text{and} \quad K_{Bm} = 1,$$

$$\theta = \frac{WL}{12(4+2)} = 5560,$$

$$M_{AB} = +\frac{400{,}000}{12} - 2(5560) = 33{,}330 - 11{,}120 = 22{,}210\text{#}',$$

$$M_{C'} = \frac{WL}{8} - M_{AB} = 50{,}000 - 22{,}210 = 27{,}790\text{#}'.$$

Therefore in this case where the column stiffness is much less than in the preceding case

$$M_{C}' = \frac{1}{8}\frac{27.79}{50.0}WL = \frac{WL}{14.4} \text{ (about).}$$

It should be noted that in the two preceding cases if the dead load on the two adjacent spans is considered the moment coefficients of $\frac{1}{18}$ and $\frac{1}{14.4}$ respectively, will become considerably smaller. Let us consider the latter case to determine the change effected in the coefficient of $\frac{1}{14.4}$ by making the dead load equal to one-third of the total load or D.L. = $\frac{1}{2}$ L.L.

$$M_{AC} \text{ will then change to } -\frac{1}{12}\left(\frac{W}{3}\right)L - 2K_{Bm}\theta,$$

and

$$\theta = \frac{\frac{WL}{12} - \frac{WL}{36}}{4(K_{Bm} + K_{Col})} = \frac{WL}{72(1.5)} = \frac{400{,}000}{108} = 3700,$$

$$M_{AB} = \frac{WL}{12} - 2K_{Bm}\theta = \frac{400{,}000}{12} - 2 \times 3700 = 25{,}900\text{#}',$$

$$M_{C'} = \frac{WL}{8} - M_{AB} = 50{,}000 - 25{,}900 = 24{,}100\text{#}',$$

and the positive moment coefficient for this case becomes

$$\frac{1}{8}\frac{24.1}{50.0} = \frac{1}{16.6}.$$

When $K_{Col} = 2$ instead of $\frac{1}{2}$ and D.L. $= \frac{1}{2}$ L.L. is on adjacent spans, the coefficient of $\frac{1}{18}$ changes to

$$\frac{1}{8}\left(\frac{20.4}{50}\right) = \frac{1}{19.6}.$$

Let us finally consider the case where the stiffness of the column disappears and we get the bearing wall case with little or no restraint to the beam as it passes over supports. In this case the three-moment equation has the most direct application. We will use the form

$$-M_1 - 4M_2 - M_3 = \frac{W_1L_1}{4} + \frac{W_2L_2}{4},$$

since all beams are assumed to have same I and L.

$M_1 = M_2 = M_3, \ldots$ practically, if we assume the interior condition indefinite in extent; this corresponds to the previous assumption that $\theta_A = \theta_C = \theta$. Also assuming D.L. $= \frac{1}{2}$ L.L. and $W =$ D.L. + L.L. as before

$$6M_A = -\frac{WL}{4} - \frac{1}{4}\left(\frac{W}{3}\right)L,$$

$$M_A = -\frac{WL}{18},$$

$$M_{C'} = (\tfrac{1}{8} - \tfrac{1}{18})WL = \frac{1}{14.4}WL.$$

For the case just discussed, of no column restraint, if the live load is very large compared with the dead load, we get the extreme value for positive moment at the center of an interior span. For live load only on the span in question with no load on adjacent spans (to which this case approaches) we have again

$$6M_A = -\frac{WL}{4},$$

and

$$M_A = -\frac{WL}{24},$$

whence

$$M_{C'} = \frac{WL}{12}.$$

It should be noted that in reinforced concrete construction the condition which produces a positive center moment of $\frac{1}{12}WL$ very seldom occurs and the recommendation of Joint Committee for a $\frac{1}{16}$ coefficient is really on the safe side for all the cases to which it applies at all.

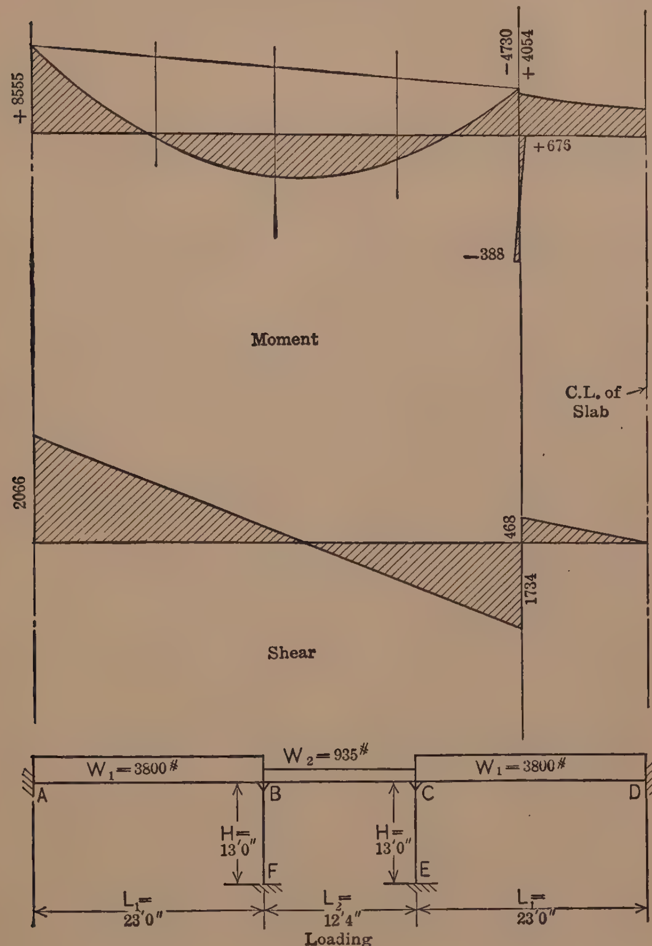


Fig. 124

Moment Equations

$M_{AB} = M_{DC}$	$= - K_{B1}(2\theta_A + \theta_B) + M_{FAB}$
$M_{BA} = M_{CD}$	$- K_{B1}(2\theta_B + \theta_A) - M_{FBA}$
$M_{BF} = M_{CE}$	$- K_C(2\theta_B + \theta_F)$
$M_{BC} = M_{CB}$	$- K_{B2}(2\theta_B + \theta_C) + M_{FBC}$
ΣM_B	$- (2K_{B1} + K_C + 2K_{B2})\theta_B - K_{B2}\theta_C - M_{FBA} + M_{FBC}$
$M_{FB} = M_{EC}$	$- K_C(2\theta_F + \theta_B)$

Joint	$\theta_A = \theta_D$	$\theta_B = \theta_C$	$\theta_F = \theta_E$	Fixed Beam Moments
$B = C$	0	$- 2K_{B1} - K_C$ $- 3K_{B2}$	0	$- M_{FBA} + M_{FBC}$

$$M_{FAB} = M_{FBA} = \frac{W_1L_1}{12} = \pm 7280'\#$$

$$M_{FBC} = M_{FCB} = \frac{W_2L_2}{12} = \pm 963'\#$$

$$K_C = \frac{I_C}{H} = 1.00$$

$$K_{B1} = K_C \times \frac{I_{B1}}{L_1} = 3.77$$

$$K_{B2} = K_C \times \frac{I_{B2}}{L_2} = 3.05.$$

SUBSTITUTING NUMERICAL VALUES

$B = C$ θ_B		-17.69 $+338$		-6327

MOMENT VALUES

M_{AB}	3.77 (338) + 7280	+ 8555
M_{BA}	3.77 (676) − 7280	− 4730
M_{BF}		+ 676
M_{BC}	3.05 (676 + 963	+ 4054
M_{FB}		+ 388

Many of the leading architects and engineers are getting away from the customary conservative value of $\frac{1}{12}WL$ for positive moments in concrete frame construction and are following the lead of the Joint Committee recommendations.

The results of the procedure discussed are summarized in Table VIII.

TABLE VIII

Coefficients of WL for Max. Positive Moments			
Condition of Loading	$k_{col} = 2k_{BM}$	$k_{col} = \frac{1}{2}k_{BM}$	Non-restraining Supports
L.L. only	$+\frac{1}{18}$	$+\frac{1}{14.4}$	$+\frac{1}{12}$
L.L. + D.L. (D.L. $= \frac{1}{2}$L.L.)	$+\frac{1}{19.6}$	$+\frac{1}{16.6}$	$+\frac{1}{14.4}$

Problems illustrated in Figs. 124 and 125 taken from conditions in a certain large building recently erected will afford the student examples of how column restraint affects the distribution of bending moments in beams, and will serve to illustrate how in many cases the standard approximate coefficients are of little use.

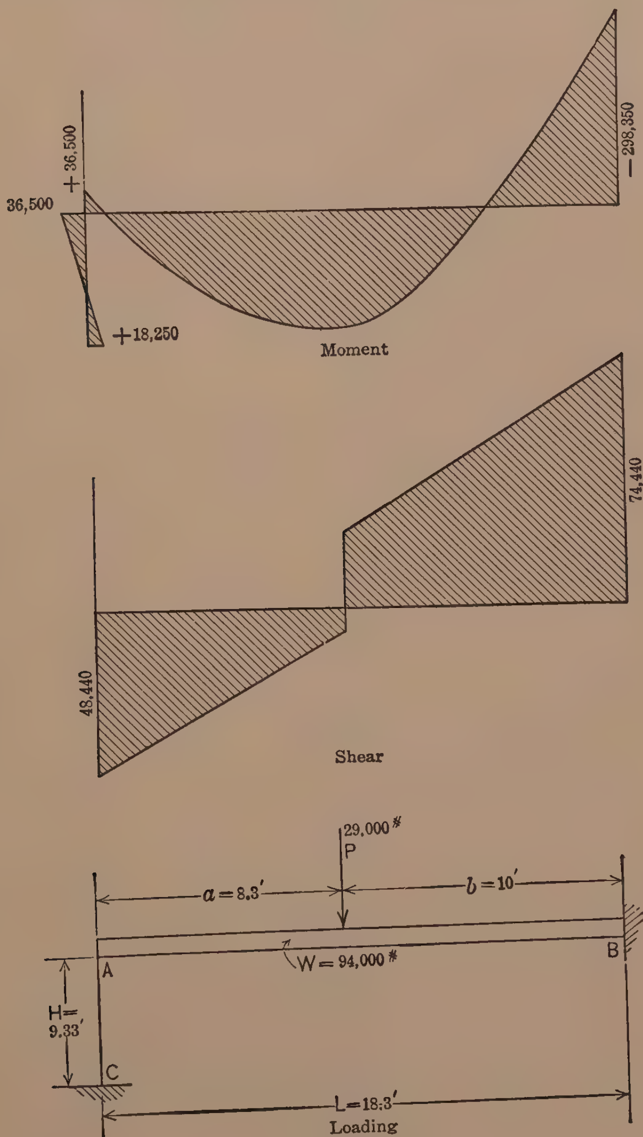


FIG. 125

MOMENT EQUATIONS

M_{AC}	$= - K_C(2\theta_A + \theta_C)$
M_{AB}	$- K_B(2\theta_A + \theta_B) + M_{FAB}$
ΣM_A	$- (2K_C + 2K_B)\theta_A + M_{FAB}$
M_{CA}	$- K_C(2\theta_C + \theta_A)$
M_{BA}	$- K_B(2\theta_B + \theta_A) - M_{FBA}$

Joint	θ_A	θ_B	θ_C	Fixed Beam Moments
A	$-2K_C - 2K_B$	0	0	$+ M_{FAB}$

$$M_{FAB} = 218{,}100'\#$$

$$M_{FBA} = 207{,}300'\#$$

$$K_C = \frac{I_C}{H} = 1.00.$$

$$K_B = K_C \times \frac{I_B}{L} = 4.97.$$

SUBSTITUTING NUMERICAL VALUES

A θ_A	-11.94 $-18{,}250$			$+218{,}100$

MOMENT VALUES

M_{AC}		$-36{,}500$
M_{AB}	$4.97 \times -36{,}500 + 218{,}100$	$+36{,}500$
M_{CA}		$-18{,}250$
M_{BA}	$4.97 \times -36{,}500 - 207{,}300$	$-298{,}350$

96. Case of Combined Translation and Rotation of the Joints of a Framed Bent.—We have thus far treated only cases of frames where no relative translation of the joints occurs. We shall now investigate the case where the forces are so applied that we do have a relative displacement of the joints. We may illustrate the relation of the two cases by the simple problems illustrated in Figs. 126 and 127.

A frame loaded as in Fig. 126, will show a slight transverse swing of the columns unless the load P is at the center, or unless the top of the frame is restrained from horizontal movement (as would many times be the case). If this condition holds, the problem is completely solved as soon as we know Θ_B and Θ_C (see Table A), but otherwise we have the additional unknown angular rotation $R\left(=\frac{D}{L}\right)$ of the line joining of the ends of the columns.

If we assume (as is almost always permissible) that the axial deformation of BC is negligible, we shall have $D_B = D_C$ and $R_{BA} = R_{CD}$ and the solution will consist in obtaining the three unknowns from three equations—the equilibrium equations for the joints B and C, which we

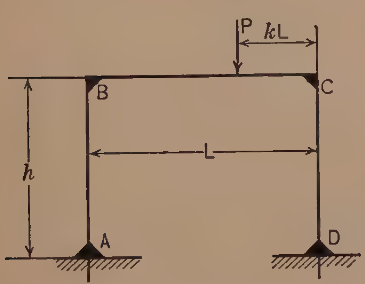


FIG. 126

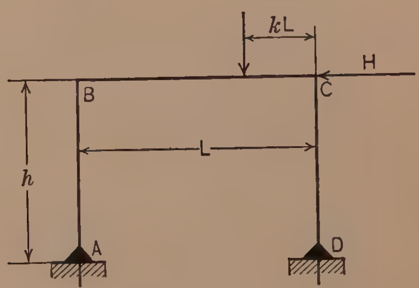


FIG. 127

TABLE A

Equation	θ_B	θ_C	Right-hand Member
J_B	$4(K_{BA} + K_{BC})$	$2K_{BC}$	$= + PL(k^2 - k^3)$
J_C	$2K_{BC}$	$4(K_{CD} + K_{BC})$	$= - PL(k - 2k^2 + k^3)$

have designated as the "joint equations," and a third statical equation expressing the fact that the sum of the resisting moments at the tops and bottoms of the columns must equal the shear $\times$ the column height $(= Hh)$, which we call the "bent equation." A convenient general form for this equation may be written as follows:

Let H = shear transferred by bending at tops and bottoms of bent-columns members, then

$$H = \sum \frac{M_{TC} + M_{BC}}{L_C},$$

where M_{TC} = moment at top of bent column $= 2K_C(3R - 2\Theta_{TC} - \Theta_{BC})$;

M_{BC} = moment at bottom of bent column $= 2K_C(3R - 2\Theta_{BC} - \Theta_{TC})$;

L_C = length of bent column;

K = *relative* $\frac{I}{L}$ value for column in question;

R = rotation of line between ends to translation of one end (plus (+) if clockwise);

Θ_{TC} = joint rotation at top of column;

Θ_{BC} = joint rotation at bottom of column.

$$\therefore\ H = \sum \frac{6K(2R - \Theta_{TC} - \Theta_{BC})}{L_C}. \quad . \quad . \quad . \quad . \quad (46)$$

It is well, again, to emphasize the sign convention as given in Chapter III. All signs of M are (+) when they act on the joint in such a manner as to turn it clockwise, and (−) if counter-clockwise. Also both R and Θ are (+) or (−) according as the sense of the rotation is clockwise or counter-clockwise from the original position.

In order to bring out clearly the differences between the two cases we will note the solution for the case of Fig. 126, where it is assumed that there is sufficient horizontal restraint at the top of the frame to maintain the lines BA and CD vertical. We then have $R = 0$ and all the moments may be obtained from Θ_B and Θ_C. The detail is very simple; we have:

$$M_{BA} + M_{BC} = 0; \quad M_{CD} + M_{CB} = 0,$$

$$M_{BA} = -\ 2K_{BA}(2\Theta_B),$$

$$M_{BC} = -\ 2K_{BC}(2\Theta_B + \Theta_C) + M_{FBC},$$

$$M_{CB} = -\ 2K_{BC}(2\Theta_C + \Theta_B) + M_{FCB},$$

$$M_{CD} = -\ 2K_{CD}(2\Theta_C),$$

$$M_{FBC} = PL(k^2 - k^3)$$

and

$$M_{FCB} = -\ PL(k - 2k^2 + k^3).$$

Then using K_{BA}, K_{BC} and K_{CD} as relative values for the stiffness of the members, and obtaining numerical values for the constant terms (end moments for BC considered as a fully restrained beam) we may solve for values of Θ_B and Θ_C which when substituted back in the original

expression for M, will give the correct moment values. Table A shows the tabulation of the equations.

96a. Illustrative Numerical Problem:

Let $K_{BA} = K_{CD} = 1$.

Then if the member BC has the same cross section as the vertical members, but is only two-thirds as long, K_{BC} (which represents and $\frac{I}{L}$ value relative to K_{BA} or unity) will be $\frac{1}{\frac{2}{3}}$ or 1.5.

$$
\begin{aligned}
\text{Let } L_{BC} &= 10', \quad L_{AB} = 15' \\
P &= 1000\# \\
k &= \tfrac{1}{4} \text{ (measured from } C) \\
K_{BC} &= 1.5 \quad \text{and} \quad K_{AB} = K_{CD} = 1.
\end{aligned}
$$

Therefore

$$M_{FBC} = 1000(10)(\tfrac{1^2}{4} - \tfrac{1^3}{4}) = \frac{30{,}000}{64} = 469\#'$$

$$M_{FCB} = -1000(10)[\tfrac{1}{4} - 2(\tfrac{1}{4})^2 + \tfrac{1^3}{4}] = -\frac{90{,}000}{64} = -1408\#'.$$

Substituting in the general equations (see table A) and solving we get:

$$
\begin{aligned}
10\theta_B + 3\theta_C &= +469, \\
3\theta_B + 10\theta_C &= -1408, \\
\theta_B &= -170.3, \\
\theta_C &= +98.0,
\end{aligned}
$$

$$
\left.
\begin{aligned}
M_{BA} &= -2(2 \times 98.0) = -392 \\
M_{BC} &= -2 \times 1.5[2 \times 98 + (-170.3)] \\
&\qquad + 469 = +392
\end{aligned}
\right\} \therefore \Sigma M_B = 0, \text{ check.}
$$

$$
\left.
\begin{aligned}
M_{CB} &= -2 \times 1.5[2 \times (-170.3) + 98.0] \\
&\qquad -1408 = -680 \\
M_{CD} &= -2[2(-170.3)] = \qquad +681
\end{aligned}
\right\} \therefore \Sigma M_C = 0, \text{ check.}
$$

$$
\begin{aligned}
M_{AB} &= -2(98.0) = -196, \\
M_{DC} &= -2(-170.3) = 341
\end{aligned}
$$

96b. Horizontal Load at C Causing Translation of Joints B and C Relative to A and D.—We shall now solve the same frame when loaded with a horizontal load at C in addition to the vertical load P. We then have the additional distortion R, and the additional statical equa-

tion (bent equation) previously indicated, i.e., that ΣV at top or bottom of columns equals the sum of the moments at the column ends. We shall take the horizontal load = 200# as indicated in Fig. 128.

We have:

$$\Sigma V = \frac{\Sigma M}{h} = \frac{2K(12R - 3\theta_B - 3\theta_C)}{h} = 200,$$

or

$$4KR - K(\theta_B)K(\theta_C) = \frac{200h}{6}.$$

$$\Sigma M = \Sigma \text{ moments top and bottom of vertical members},$$
$$h = \text{height of vertical member},$$
$$K = \frac{I}{L} \text{ for vertical members.}$$

As before

$$\Sigma M_B = 0 \quad \text{and} \quad \Sigma M_C = 0,$$

whence

$$M_{BA} + M_{BC} = 0,$$

and

$$M_{CD} + M_{CB} = 0,$$

but the new unknown R becomes a factor in the expression for moments in the columns, and we now have

$$M_{BA} = +\,2K_{AB}(3R - 2\theta_B),$$
$$M_{BC} = -\,2K_{BC}(2\theta_B + \theta_C) + M_{FBC},$$
$$M_{CB} = -\,2K_{BC}(2\theta_C + \theta_B) + M_{FCB},$$
$$M_{CD} = 2K_{CD}(3R - 2\theta_C).$$

Expressing the bent equation and the two joint equations in form for solution we have the results shown in Table A, page 236.

Substituting back in the moment equations,

$$M_{BA} = 2[3(139.0) - 2 \times 162] = 186.0,$$
$$M_{BC} = -\,2 \times 1.5[2 \times 162 + (-\,106.2)] + 469 = -\,184.4.$$
$$\therefore\ \Sigma M_B = 0, \text{ check.}$$
$$M_{CB} = -\,2 \times 1.5[2(-\,106.2) + 162.0] - 1408 = -\,1256.8,$$
$$M_{CD} = +\,2[3(139.0) - 2(-\,106.2)] = +\,1258.8.$$
$$\therefore\ \Sigma M_C = 0, \text{ check.}$$
$$M_{AB} = +\,2[3(139.0) - 162.0] = 510,$$
$$M_{DC} = +\,2[3(139.0) - (-\,106.2)] = 1046.4.$$

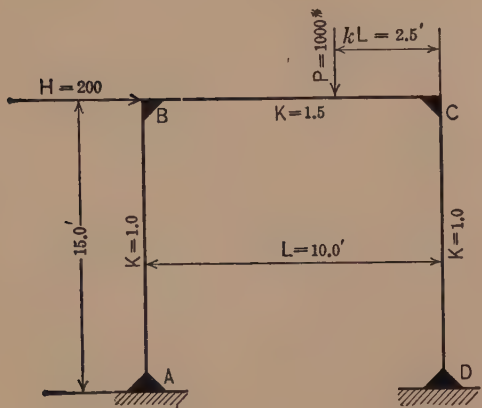


FIG. 128

TABLE B

Solution of Equations and Moment Calculations for Bent with Unsymmetrical Vertical Load Only and no Horizontal Restraint (Fig. 126)

Equation	R	θ_B	θ_C	Constant Term	Moments
Bent	+24	− 6	− 6	0	$M_{BA} = 2[3(-23.4) - 2(+86.9)] = -488$
Joint B	− 6	10	3	+ 469'#	
Joint C	− 6	3	10	−1408'#	$M_{BC} = -2 \times 1.5[2(+86.9) + (-180.5)] = +489$
1	1.00	− .25	− .25	0	
2	−1.00	+1.67	+ .50	+ 78.2	$M_{CB} = -2 \times 1.5[2(-180.5) + 86.9] = -586$
3	−1.00	+ .50	+1.67	−234.7	
1 + 2	......	+1.42	+ .25	+ 78.2	$M_{CD} = 2[3(-23.4) - 2(-180.5)] = +582$
1 + 3	......	+ .25	+1.42	−234.7	
4	......	+1.00	.176	+ 55.1	$M_{AB} = 2[3(-23.4) - (+86.9)] = -314$
5	......	+1.00	5.680	−938.8	
5 − 4	......	......	5.504	−993.9	$M_{DC} = 2[3(-23.4) - (-180.5)] = +220$
....	......	......	θ_C =	−180.5	
From 4	......	......	θ_B =	+ 86.9	
From 1	......	......	R =	− 23.4	

TABLE A

Solution of Equations for Bent with Horizontal and Vertical Loads (Fig. 128)

Equation	Unknowns			Constant Term
	θ_B	θ_C	R	
J_B	$4(K_{BA} + K_{BC})$	$2K_{BC}$	$-6K_{AB}$	$+PL(k^2 - k^3)$
J_C	$2K_{BC}$	$4(K_{CD} + K_{BC})$	$-6K_{CD}$	$-PL(k - 2k^2 + k^3)$
Bent	$-K_{AB}$	$-K_{AB}$	$+4K_{AB}$	$Hh/6$
J_B	1.0	3	-6	$+469$
J_C	3	1.0	-6	-1408
Bent	-1	-1	$+4$	$+500$
①	1.0	.300	$-$ .600	$+469$
②	1.0	3.333	-2.000	-469.3
③	1.0	1.000	-4.006	-500.0
		-3.033	$+1.40$	$+516.2$
		$+2.333$	$+2.00$	$+$ 30.7
		1	$-$ .462	-170.5
		1	$+$ .858	$+$ 13.2
			-1.320	-183.7
			R =	$+139.0$
			θ_C =	-106.2
			θ_B =	$+162.0$

Further

$$\frac{M_{BA} + M_{AB}}{h} + \frac{M_{CD} + M_{DC}}{h} = V_{BA} + V_{CD}$$

$$= \frac{186 + 510}{15} + \frac{1258 + 1046}{15} = 200,$$

the applied shear, therefore bent equation is also satisfied.

In Fig. 129 the moments as found in the preceding example are plotted and resulting transverse shears in the members are shown together with a distortion sketch indicating Θ and R values.

It may be interesting to note the solution for the frame when only the load P is applied, but with no horizontal restraint at the top. This case will also involve a transverse swing as noted previously. The complete solution is shown in Table B, page 235.

97. The Framed Bent with Inclined Legs.—One of the simpler cases of the framed bent with inclined side members is that of the so-called "A" frame. To illustrate the method of attack for such a problem, we will take the case of a frame similar in dimensions and loading to

that used as an example in the preceding article but with inclined instead of vertical side members. (See Fig. 130.)

The horizontal movement of B and $C = Rh$ as before,
The vertical *drop* of $B = RL_1$,
The vertical *lift* of $C = RL_1$.

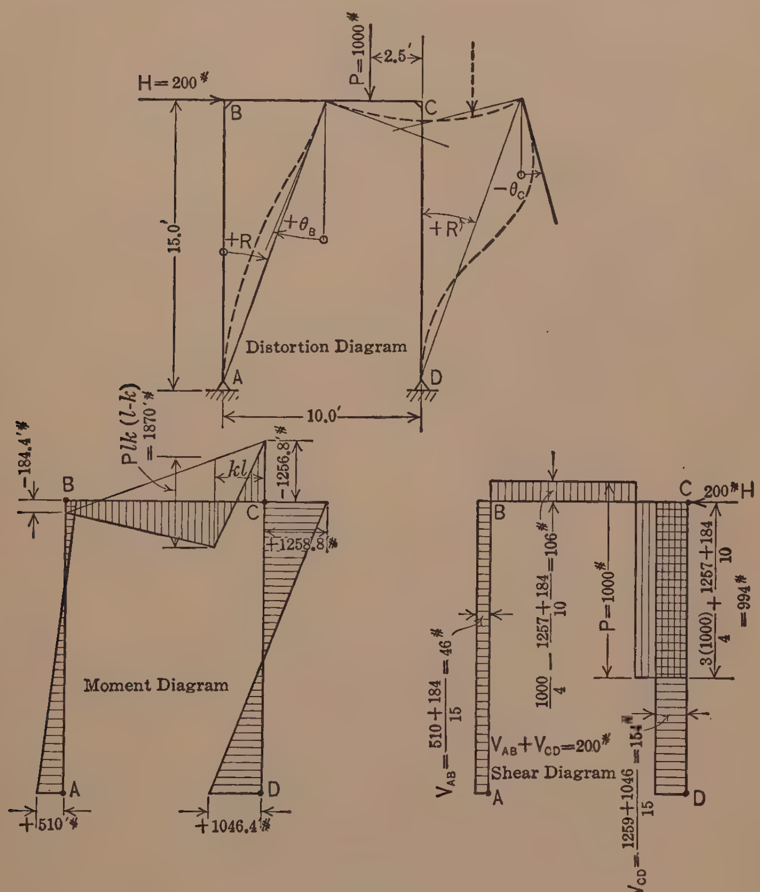


FIG. 129

Therefore the member BC undergoes a rotation of $R\frac{2L_1}{L}$ which in this case equals R (since $2L_1 = L$) and which is $(-)$ or counter-clockwise with the load P acting from the left.

We shall have for simultaneous solution the same two joint equa-

tions and the same bent equation as before. The expressions for the moments are:

$$M_{AB} = 2K_{AB}(3R_{AB} - \theta_B),$$
$$M_{BA} = 2K_{AB}(3R_{AB} - 2\theta_B),$$
$$M_{BC} = 2K_{BC}(-3R_{BC} - 2\theta_B - \theta_C) + M_{FBC},$$
$$M_{CB} = 2K_{BC}(-3R_{BC} - 2\theta_C - \theta_B) + M_{FCB},$$
$$M_{CD} = 2K_{CD}(3R_{CD} - 2\theta_C).$$

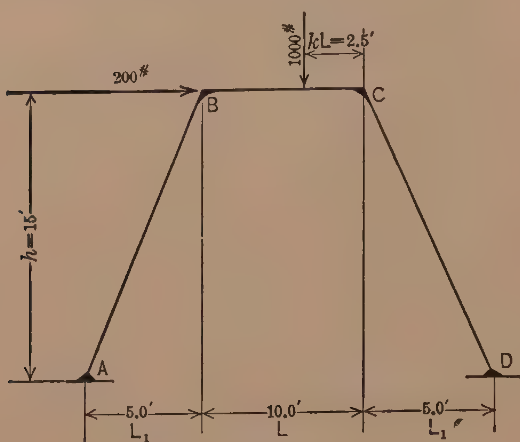


Fig. 130

Also:

$$\left.\begin{array}{l} (a)\; M_{BA} + M_{BC} = 0 \\ (b)\; M_{CB} + M_{CD} = 0 \end{array}\right\} \ldots \text{Joint Equations,}$$

$$(c)\; M_{AB} + M_{BA} + M_{CD} + M_{DC} = Hh \ldots \text{Bent Equation,}$$
$$R_{AB} = R_{CD} = -R_{BC} \text{ (from preceding paragraph),}$$

therefore we have the three equations indicated for solution for the three unknowns, R, θ_B, θ_C.

Substituting numerical values for M_F and K,

$$K_{AB} = K_{CD} = 1.0; \quad K_{BC} = 1.5,$$
$$M_{FBC} = 469\,\#'; \quad M_{FCB} = -1408\,\#',$$

equations (a), (b) and (c) become,

$$-3R - 10\theta_B - 3\theta_C + 469 = 0 \; (a').$$
$$-3R - 10\theta_C - 3\theta_B - 1408 = 0 \; (b').$$
$$2(6R - 3\theta_B + 6R - 3\theta_C) = -Hh,$$

or

$$4R - \theta_B - \theta_C = \frac{Hh}{6} = 500 \; (c').$$

Solving these three simultaneous equations in tabular form we have the following:

TABLE A

Equation	θ_B	θ_C	R	Right-hand Member
J_B	10	3	3	469
J_C	3	10	3	−1408
Bent	−1	−1	4	+500
J_B	1.0	.30	.30	+46.9
J_C	1.0	3.33	1.00	−469
Bent	1.0	1.0	−4.00	−500
		3.03	.70	−515.9
		2.33	5.00	+31.0
		1.0	.231	−170.2
		1.0	2.145	+13.3
			1.914	+183.5
				$R =$ +95.8
				$\theta_C =$ −192.3
				$\theta_B =$ +75.8

$$M_{AB} = 2[3(95.8) - 75.8] = 423.2.$$

$$M_{BA} = 2[3(95.8) - 2(75.8)] = \ldots + 271.6,$$

$$M_{BC} = 3[-3(95.8) - 2(75.8) - (-192.3)] + 469 = -271.1,$$

check.

$$M_{CB} = 3[-3(95.8) - 2(-192.3) - 75.8] - 1408 = -1343.8,$$

$$M_{CD} = 2[3(95.8) - 2(-192.3)] = \ldots 1344.0,$$

check.

$$M_{DC} = 2[3(95.8) - (-192.3] = 959.4.$$

It will be seen from the preceding that the two joint equations are satisfied and that for the bent equation

$$M_{BA} + M_{AB} + M_{CD} + M_{DC} = 423.2 + 271.6 + 1344 + 959.4,$$

$$\left.\begin{aligned} &= 2998.2 \\ Hh &= 3000 \end{aligned}\right\} \text{check.}$$

When compared with a similar problem where the posts are vertical it will be seen that the inclination of the posts gives the effect of a slightly greater stiffening tendency from the transverse member. The horizontal deflection of the top for a given applied horizontal load will be less where the posts are inclined and the calculations show that the inclination of post throws larger moments into the ends of the horizontal member and into the tops of the posts, which would necessarily accompany a greater stiffening effect of the horizontal member.

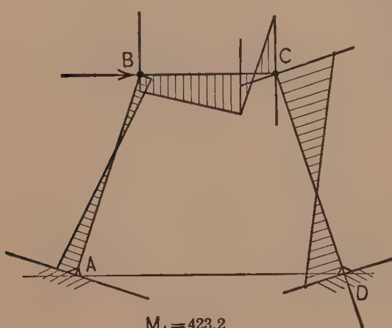


FIG. 131

A comparison of the moment diagram here given for the present case with that of the preceding example will serve to illustrate these statements.

The same methods here used for the one-story bent with inclined side members will apply to bents of two or more stories. It should be noted that the R value for the horizontal member introduces no *new* unknown and can always be shown to be a definite function of the R value for the side members,—the latter being determined by the bent equation.

98. The Rectangular Bent with Transverse Loading and Columns of Different Lengths.—(Fig. 132). This is the typical problem of the reinforced concrete bridge bent or special 2-span culvert bridge. The problem is complicated by the fact that the vertical supporting members have varying lengths. As regards simplicity of solution and number of unknowns the problem is the same as if all columns were of the same length. If we substitute for R its value, $\frac{D}{H}$, D being the horizontal deflection of the tops of the columns (assumed the same for all points), the unknowns to solve for become Θ_A, Θ_B, Θ_C and D.

Assuming D, E, and F fixed we have the three joint equations and one bent equation necessary to solve for the ten bending moments (all different) at each end of the five different members which compose the bent. The moments are

$$M_{AD} = 2K_{AD}\left(3\frac{D}{H_{AD}} - 2\Theta_A\right) = \ldots + 4(\tfrac{3}{10}D - 2\Theta_A),$$

$$M_{AB} = MF_{AB} - 2K_{AB}(2\Theta_A + \Theta_B) = + \frac{2000(20)^2}{12} - 4(2\Theta_A + \Theta_B),$$

$$M_{BA} = -M_{FBA} - 2K_{AB}(2\Theta_B + \Theta_A) = -66{,}700 - 4(2\Theta_B + \Theta_A),$$

$$M_{BC} = M_{FBC} - 2K_{BC}(2\Theta_B + \Theta_C) = +\frac{2000(25)^2}{12} - 4(2\Theta_B + \Theta_A),$$

$$M_{BE} = 2K_{BE}\left(3\frac{D}{H_{BE}} - 2\Theta_B\right) = \ldots 2.8(\tfrac{3}{15}D - 2\Theta_B),$$

$$M_{CB} = -M_{FCB} - 2K_{BC}(2\Theta_C + \Theta_B) = -104{,}000 - 4(2\Theta_C + \Theta_B),$$

$$M_{CF} = 2K_{CF}\left(3\frac{D}{H_{CF}} - 2\Theta_C\right) = \ldots 2(\tfrac{3}{20}D - 2\Theta_C),$$

$$M_{DA} = 2K_{AD}\left(3\frac{D}{H_{AD}} - \Theta_A\right) = \ldots 4(\tfrac{3}{10}D - \Theta_A),$$

$$M_{EB} = 2K_{BE}\left(3\frac{D}{H_{BE}} - \Theta_B\right) = \ldots 2.8(\tfrac{3}{15}D - \Theta_B),$$

$$M_{FC} = 2K_{CF}\left(3\frac{D}{H_{CF}} - \Theta_C\right) = \ldots 2(\tfrac{3}{20}D - \Theta_C).$$

From these we set up the equations for the Θ's and D

$$\left.\begin{aligned} (a)\ \Sigma M_A &= 0 = -16\Theta_A - 4\Theta_B + 1.2\Theta D + 66{,}700 \\ (b)\ \Sigma M_B &= 0 = -4\Theta_A - 21.6\Theta_B - 4\Theta_C + .56D + 37{,}300 \\ (c)\ \Sigma M_C &= 0 = -4\Theta_B - 12\Theta_C + .30D - 104{,}000 \end{aligned}\right\} \text{Joint Equations.}$$

$$(d)\ \frac{M_{AD} + M_{DA}}{10} + \frac{M_{BE} + M_{EB}}{15} + \frac{M_{CF} + M_{FC}}{20} = -18{,}000,$$

or, substituting and combining,

$$-1.2\Theta_A - .56\Theta_B - .30\Theta_C + .345D = -400 \ldots \text{Bent Equation.}$$

The minus sign is given to 400, the shearing force on the bent, because its tendency is to cause all columns to rotate in a *counter-clockwise* direction.

In Table A we have a solution of these equations, a substitution in the original moment equations and a check of the three joint equation and the bent equation.

Fig. 132 gives the moment diagram, shear diagram, and also a distortion sketch.

99. The Framed Bridge Span or Open Webbed Girder with the Loading Applied between Joints.—The method of attack for such a problem can best be illustrated by the carrying through of the numerical

solution of a specific case. Where the problem involves the solution of a number of simultaneous equations, the attempt to carry through a general case (without assigning numerical values) is exceedingly labori-

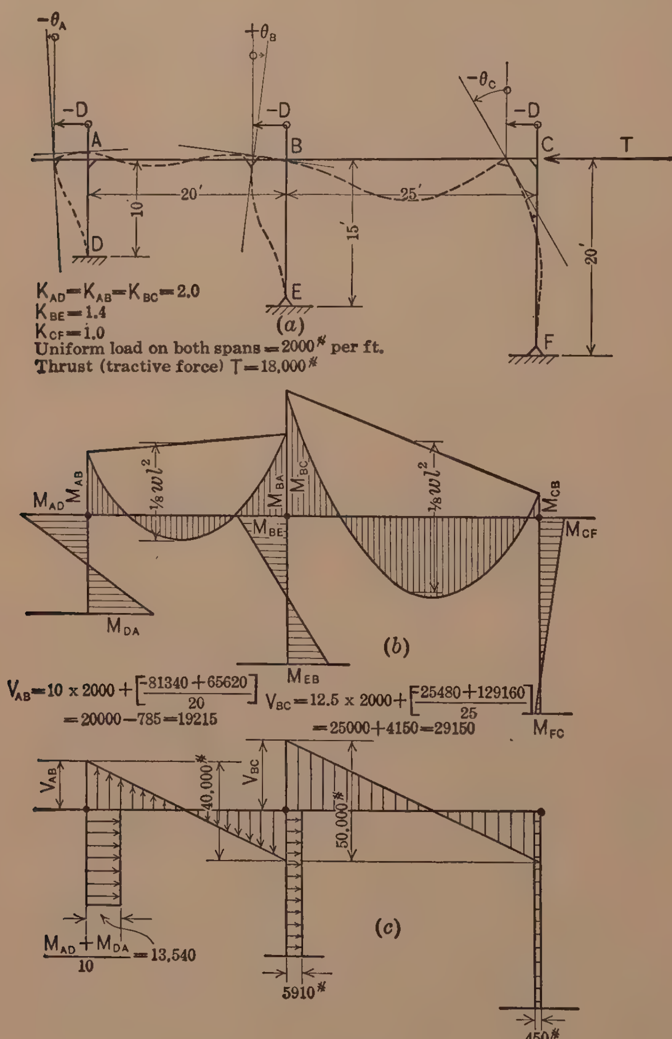


FIG. 132

ous and results in expressions so complicated as to be of little practical use. Further, the fundamental principles are often lost sight of in such a solution.

TABLE A

Rectangular Bent with Columns of Different Lengths (Fig. 132)

Equation	Unknowns to Be Solved for				Constant Term of Equation
	θ_A	θ_B	θ_C	D	
J_A	$+16$	$+4$	....	-1.2	$+66,700$
J_B	$+4$	$+21.6$	$+4$	$-.56$	$+37,300$
J_C	....	$+4$	$+12$	$-.30$	$-104,000$
Bent	-1.2	$-.56$	$-.30$	$+.345$	$-18,000$
J_A	$+1.0$	$+.250$	....	$-.075$	$+4170$
J_B	$+1.0$	$+5.400$	$+1.00$	$-.140$	$+9320$
Bent	$+1.0$	$+.466$	$+.25$	$-.287$	$+15,000$
$J_A - J_B$	....	-5.150	-1.00	$+.065$	-5150
$J_B -$ Bent	....	$+4.934$	$+.75$	$+.147$	-5680
1	....	1.00	$+.194$	$-.0126$	$+1000$
2	....	1.00	$+.152$	$+.0298$	-1152
J_C	....	1.00	$+3.00$	$-.075$	$-26,000$
$1 - 2$	....	....	$+.042$	$-.0424$	$+2152$
$J_C - 2$	....	....	$+2.848$	$-.1048$	$-24,848$
3	....	....	$+1.00$	-1.0100	$+51,200$
4	....	....	$+1.00$	$-.0367$	-8730
$3 - 4$	....	....	....	$-.9733$	$+59,930$

$$D = -61600$$

$$\theta_C = -8730 + .0367(-61600) = -10990$$

$$\theta_B = +1000 - .194(-10990) + .0126(-61600) = +2350$$

$$\theta_A = +4170 - .25(+2350) + .075(-616000) = -1040$$

$$M_{AD} = 4[.3(-61,600) - 2(-1040)] = -65,600$$
$$M_{AB} = +66,700 - 4[2(-1040) + (+2350)] = +65,620$$
$$\therefore \Sigma M_A = 0$$
$$M_{BA} = -66,700 - 4[2(+2350) + (-1040)] = -81,340$$
$$M_{BC} = +104,000 - 4[2(+2350) + (-10,990)] = +129,160$$
$$M_{BE} = 2.8[.2(-61,600) - 2(+2350)] = -47,600$$
$$\therefore \Sigma M_B = 0$$
$$M_{CB} = -104,000 - 4[2(-10,990) + (+2350)] = -25,480$$
$$M_{CF} = 2.0[.15(-61,600) - 2(-10,990)] = +25,480$$
$$\therefore \Sigma M_C = 0$$
$$M_{DA} = 4[.3(-61,600) - (-1040)] = -69,760$$
$$M_{EB} = 2.8[.2(-61,600) - (+2350)] = -41,100$$
$$M_{FC} = 2.0[.15(-61,600) - (-10,990)] = +3500$$
$$M_{AD} = \quad = -65,600$$
$$M_{BE} = \quad = -47,600$$
$$M_{CF} = \quad = +25,480$$

$$\frac{-69,760 - 65,600}{10} + \frac{-41,100 - 47,600}{15} + \frac{25,480 + 3500}{20} = -18,000.$$

$\therefore$ Bent equation is satisfied and *a* shear = 18,000 is developed in columns.

In the problem here indicated, where the loading, stiffnesses and member lengths are symmetrical about the center line of the truss, we have four unknown Θ values and one unknown R value. To determine

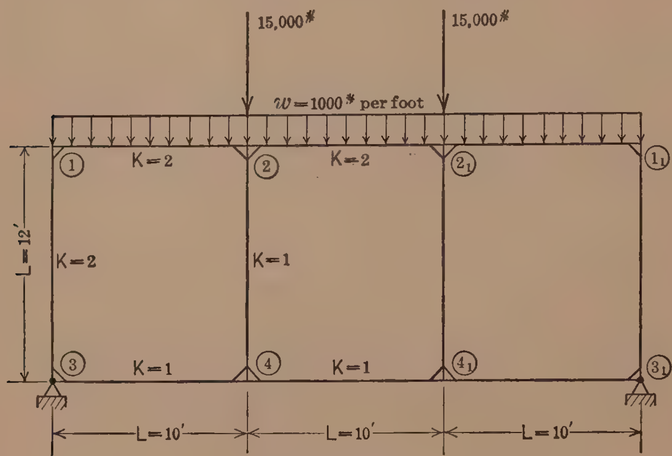


FIG. 133

these values we have four joint equations and one bent equation. The ten moment equations are as follows:

(Noting that $\Theta_2 = -\Theta_{2_1}$ and $\Theta_4 = -\Theta_{4_1}$; Reaction = 30,000),

$$M_{1-2} = 2K_{1-2}(3R - 2\Theta_1 - \Theta_2) + \frac{WL}{12} = 4(3R - 2\Theta_1 - \Theta_2) + 8330,$$

$$M_{2-1} = 2K_{1-2}(3R - 2\Theta_2 - \Theta_1) + \frac{WL}{12} = 4(3R - \Theta_1 - 2\Theta_2) - 8330,$$

$$M_{2-2_1} = 2K_{2-2_1}(-2\Theta_2 - \Theta_{2_1}) + \frac{WL}{12} = 4(-\Theta_2) + 8330,$$

$$M_{1-3} = 2K_{1-3}(-2\Theta_1 - \Theta_3)\ \ldots = 4(-2\Theta_1 - \Theta_3),$$

$$M_{3-1} = 2K_{1-3}(-2\Theta_3 - \Theta_1)\ \ldots = 4(-2\Theta_3 - \Theta_1),$$

$$M_{2-4} = 2K_{2-4}(-2\Theta_2 - \Theta_4)\ \ldots = 2(-2\Theta_2 - \Theta_4),$$

$$M_{4-2} = 2K_{4-2}(-2\Theta_4 - \Theta_2)\ \ldots = 2(-2\Theta_4 - \Theta_2),$$

$$M_{3-4} = 2K_{3-4}(3R - 2\Theta_3 - \Theta_4) = 2(3R - 2\Theta_3 - \Theta_4),$$

$$M_{4-3} = 2K_{4-3}(3R - 2\Theta_4 - \Theta_3) = 2(3R - 2\Theta_4 - \Theta_3),$$

$$M_{4-4_1} = 2K_{4-4_1}(-2\Theta_4 - \Theta_{4_1}) = 2(-\Theta_4).$$

The five equations of equilibrium are then expressed as follows:

$$\left.\begin{array}{l}
① \; M_{1-2} + M_{1-3} = 0, \;\therefore\; 12R - 16\Theta_1 - 4\Theta_2 - 4\Theta_3 + 83{,}330 = 0, \\
② \; M_{2-1} + M_{2-2_1} + M_{2-4} = 0, \;\therefore\; 6R - 8\Theta_2 - 2\Theta_1 - \Theta_4 = 0, \\
③ \; M_{3-1} + M_{3-4} = 0, \;\therefore\; 3R - 6\Theta_3 - 2\Theta_1 - \Theta_4 = 0, \\
④ \; M_{4-3} + M_{4-2} + M_{4-4_1} = 0, \;\therefore\; 3R - 5\Theta_4 - \Theta_2 - \Theta_3 = 0,
\end{array}\right\} \text{Joint Equations}$$

$$⑤ \; M_{1-2} + M_{2-1} + M_{3-4} + M_{4-3} = 30{,}000 \times 10 - 10{,}000 \times 5,$$

$$\therefore\; 18R - 6\Theta_1 - 6\Theta_2 - 3\Theta_3 - 3\Theta_4 = 125{,}000 \; . \; . \; . \text{ Bent Equation.}$$

The bent equation is obtained by cutting out a section between two vertical lines just to the left of 2–4 and just to the right of 1–3 and taking the moment of the shear on 1–3 minus moment of loads between 1–3 and 2–4 equal to resisting moments on all four member ends cut.

Solving these equations in the convenient tabular form shown in Table A, we find values of the unknowns Θ_1, Θ_2, Θ_3, Θ_4 and R.

Substituting these values back into the original moment expressions we obtain the following:

$$M_{1-2} = 4[3(+13{,}620) - 2(+8100) - 7450] + 8330 = +77{,}170,$$

$$M_{1-3} = 4[-2(8100) - 3120] = \; . \; . \; . -77{,}280$$

$$\Sigma M = 0, \text{ check.}$$

$$M_{2-1} = 4[3(+13{,}620) - (+8100) - 2(+7450)] - 8330 = 63{,}110,$$

$$M_{2-2_1} = 4[-(+7450)] + 8330 = \; . \; . \; . -21{,}470,$$

$$M_{2-4} = 2[-2(+7450) - 6060] = \; . \; . \; . -41{,}920,$$

$$\Sigma M = 0, \text{ check.}$$

$$M_{3-1} = 4[-(+8100) - 2(+3120)] = \; . \; . \; . -57{,}360,$$

$$M_{3-4} = 2[3(+13{,}620) - 2(+3120) - (+6060)] \; . \; . \; . +57{,}120,$$

$$\Sigma M = 0, \text{ check.}$$

$$M_{4-2} = 2[-(+7450) - 2(+6060)] = \; . \; . \; . -39{,}140,$$

$$M_{4-3} = 2[3(+13{,}620) - (+3120) - 2(+6060)] \; . \; . \; . +51{,}240,$$

$$M_{4-4_1} = 2[-(+6060)] = \; . \; . \; . -12{,}120,$$

$$\Sigma M = 0, \text{ check.}$$

It will be seen upon examination of these results that all the joint equations check up very closely to zero while the bent equation is also satisfied,—in other words

$$M_{1-2} + M_{2-1} + M_{3-4} + M_{4-3} = 248{,}640\text{—against } 250{,}000$$

which is an error of less than 1 per cent.

TABLE A

OPEN WEBBED GIRDER (Fig. 133)

Equations	Unknowns					Knowns
	θ_1	θ_2	θ_3	θ_4	R	
J_1	16	4	4	....	−12	+ 8330
J_2	2	8	....	1	− 6	
J_3	2	....	6	1	− 3	
J_4	....	1	1	5	− 3	
Bent	− 6	− 6	− 3	− 3	+18	+ 125,000
J_1	1.0	.250	.250	....	− .750	+ 521
J_2	1.0	4.000	....	.500	− 3.00	
J_3	1.0	....	3.00	.500	− 1.50	
Bent	− 1.0	− 1.0	− .50	− .50	+ 3.00	20,800
$J_2 - J_1$	....	3.75	− .25	.50	− 2.25	− 521
$J_2 - J_3$	....	4.00	− 3.00	....	− 1.50	
J_3 + Bent	....	− 1.00	+ 2.50	....	+ 1.50	+ 20,800
①	....	1.0	− .067	.133	− .600	− 139
②	....	1.0	− .750	....	− .375	
③	....	1.0	− 2.50	....	− 1.50	− 20,800
J_4	....	1.0	1.00	5.0	− 3.00	
① − ②	....	....	.683	.133	− .225	− 139
② − ③	....	....	1.750	....	1.125	+ 20,800
③ − J_4	....	....	− 3.50	− 5.00	+ 1.50	− 20,800
④	....	....	1.0	.195	− .329	−204
⑤	....	....	1.0	....	.643	+ 11,880
⑥	....	....	1.0	1.428	− .428	+ 5,940
④ − ⑤	....	....	....	.195	− .972	− 12,084
⑤ − ⑥	....	....	....	− 1.428	1.071	+ 5,940
⑦	....	....	....	1.0	− 4.990	− 61,900
⑧	....	....	....	1.0	− .750	− 4,160
					− 4.24	− 57,740

$R = +13{,}620$

$\theta_4 = +.75(31{,}620) - 4160$
$= +6060$

$\theta_3 = +11{,}880 - .643(13{,}620) = +3120$

$\theta_2 = .75(3120) + .375(+13{,}620) = +7450$

$\theta_1 = +.521 - .25(7450 + 3120) + .75(13{,}620) = +8100$

CHAPTER VI

THE ELASTIC ARCH

100. Preliminary.—As defined by the engineer, an arch is *any* structure which develops horizontal reactions under vertical loads. In this sense the truss of Fig. 134 is quite as definitely an arch as the curved girder of Fig. 135. As actually built, however, most arch structures

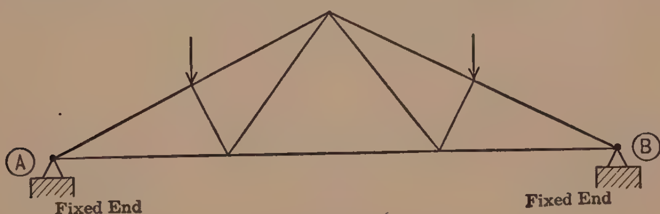


FIG. 134

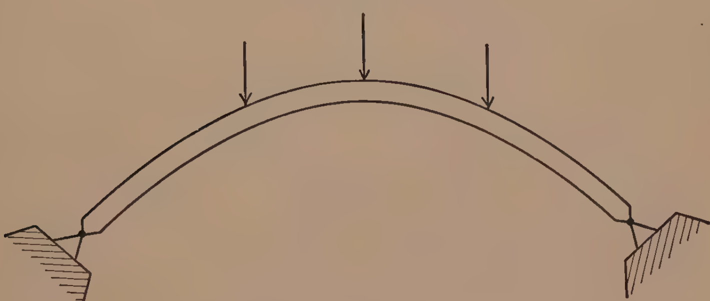


FIG. 135

have the lower chord joints and often both chord joints lying on curves convex upwards as indicated in Figs. 136 to 140 which show some typical arch structures.

The arch has a very wide range of application in bridge design. Reinforced concrete arches have been built from 30 ft. to 400 ft. spans and steel arches have been built from 200 ft. to 1000 ft. span lengths, and full designs have been prepared for much longer spans.*

* One such span designed for the North River crossing, New York, was 3100 ft. in length.

The statical advantage of arch action is illustrated by the two-hinged arch rib of Fig. 141. The large horizontal thrust developed in restrain-

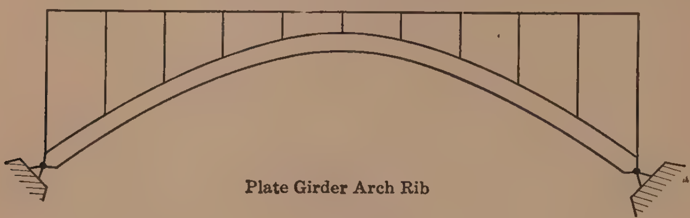


Fig. 136

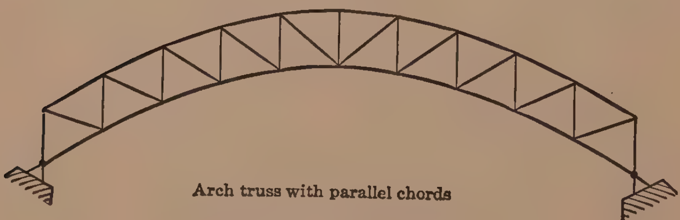


Fig. 137

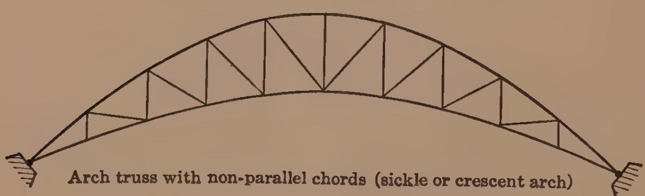


Fig. 138

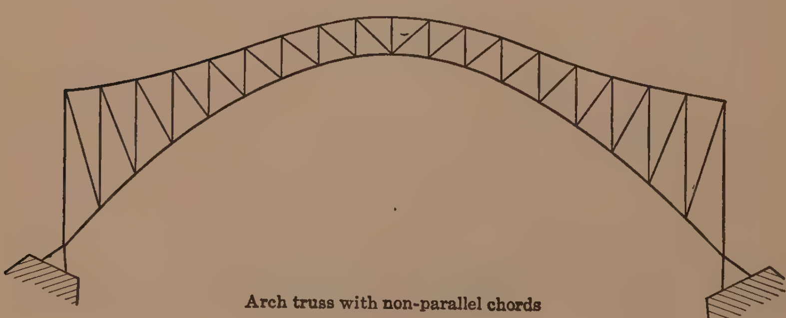
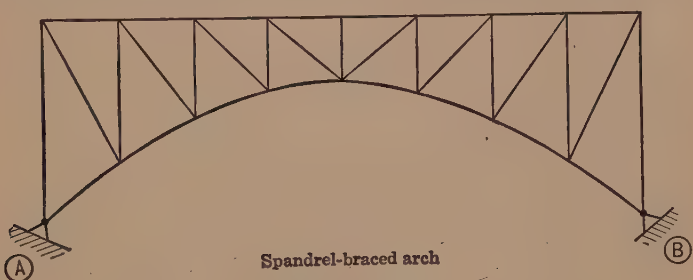


Fig. 139

ing horizontal movement induces moments tending to counteract the simple beam moments. Fig. 141*b* shows the moment diagram for the

arch (or for any other structure) acting as a simple beam. Fig. 141c shows the moment diagram due to horizontal thrust H, and Fig. 141d shows the final diagram. The great reduction of bending action is evident. As a matter of fact, if the loading is fixed, an arch may always be designed to fit the equilibrium polygon for the loads practically exactly, and in such case all bending stresses are eliminated.

The arch has other advantages—any steel arch lends itself readily to erection without falsework by the cantilever method; two-hinged and hingeless arches are relatively rigid structures, and steel arch trusses are likely to show small secondary stresses; the arch rib of steel or concrete (and some arch trusses) exhibit more graceful lines and a more pleasing appearance than a simple girder or truss, or a cantilever. Three-hinged and (less commonly) two-hinged arches are used in long span



Spandrel-braced arch

Fig. 140

roof construction, but the arch principle finds its greatest application in railway and highway bridges.

Where the crossing is over a deep gorge with rocky sides and where the stream traffic or other conditions make it impossible to erect by false-work, the arch is especially suitable, offering the double advantage of economy of material and ease of erection (see Fig. 142a and b). For any span length from perhaps 200 ft. to the limit of single arch spans (perhaps 3000 ft.) it is likely to prove advantageous for such a crossing.

It is by no means limited to such conditions (note the Hell Gate crossing for example), but it loses its peculiar advantage in proportion as the soil conditions require large increase in the masonry abutments to take up the horizontal thrust.

With such a great variety of types, the theory of arches becomes a very extensive field. We shall only consider in this Chapter the types commonly met with in American practice. These are (1) the two-hinged arch rib—either the solid rib or a relatively shallow truss with parallel chords which may be treated as a beam (see Figs. 136 and

137); (2) the two-hinged spandrel braced arch (see Fig. 140), an (3) the hingeless arch rib (solid or braced girder).

The three-hinged arch is a common structure but as it is statically determinate it will not be treated here. The one-hinged arch is almost never built in America. Two-hinged arch trusses of the type of Figs. 138 and 139, when the chords diverge sufficiently that they cannot be

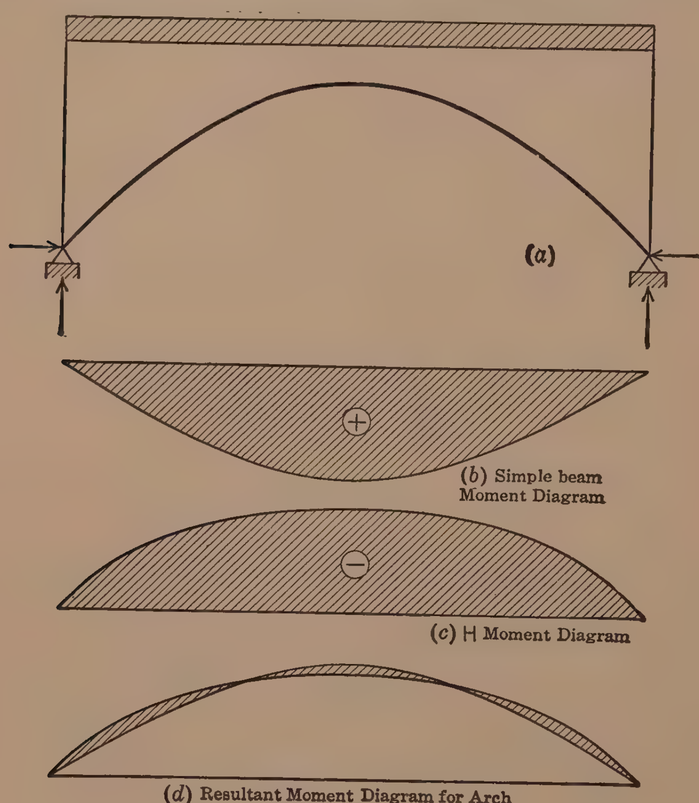


FIG. 141

treated as ribs, are analyzed on the same principle exactly as the spandrel-braced arch. The hingeless arch truss (other than the shallow-braced rib) is a very rare structure.

We shall confine our treatment in the main to symmetrical arches, though the theory presented is general and may be applied to any type of arch.

It is the purpose of the treatment to acquaint the student with the methods of analyzing the stresses in the commoner types of arch-

structures. As in the case of other indeterminate structures (for example, continuous trusses and rigid building frames) it is impossible to divorce the problem of stress analysis from the size and the make-up of the members of the structure, as may be done in the case of simple structures. Before a statically indeterminate analysis can be carried through, the cross-section properties of the constituent members must be known in addition to the loads and center line dimensions; so that the process of

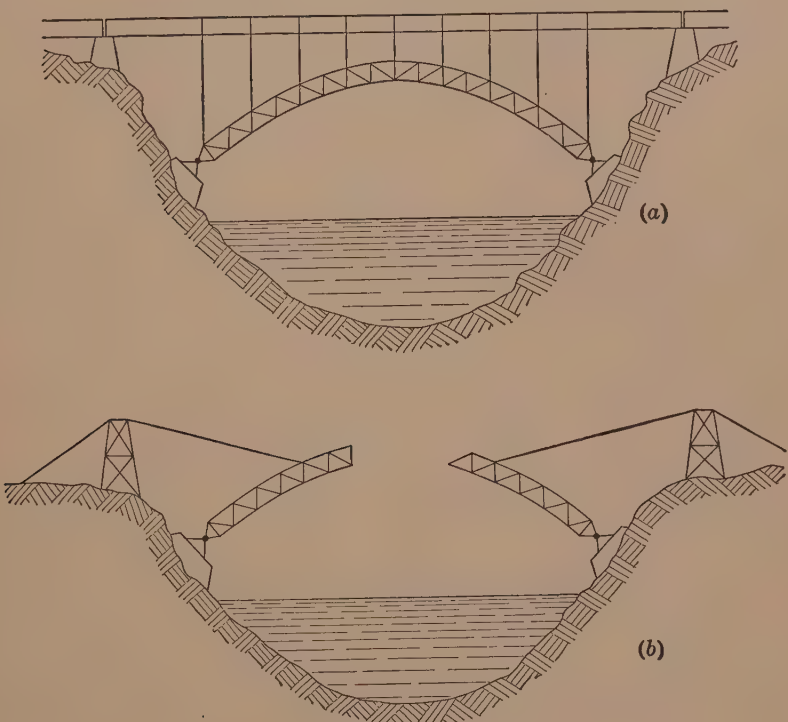


FIG. 142

design is in a manner intimately tied up with that of stress computation. The method of procedure is indicated in the problems of articles 107 and 117, but of course any treatment of the major problems of arch design is quite beyond the scope of this book. It should be emphasized that when one has at his command methods for analyzing statically indeterminate stresses, the problem of the design of a determinate or indeterminate structure is placed on the same footing, so far as a correct and scientific method of procedure is concerned. But for either type of structure, a correct analysis of the stresses is but one step in the design,

if we use the latter in its broad sense. For a discussion of the design of arches, using "design" to mean the selection of most favorable types and forms, the economical proportions of main sections and details, etc., the student must be referred to special treatises and articles.*

SECTION I.—THE TWO-HINGED ARCH

101. The General Problem.—The two-hinged arch presents a singly statically indeterminate problem and as such it has already been treated briefly as a part of the general theory in Chapter II. The horizontal reaction is usually taken as the redundant † and whatever the type of arch we shall always have the fundamental relation

$$H = -\frac{\delta'_H}{\delta_{1H}}, \qquad (47)$$

where

δ'_H = the horizontal deflection at the support when H is removed entirely, and

δ_{1H} = horizontal deflection at the support due to $H = 1$, no other loads acting.

The chief question, then, is the evaluation of the quantities δ'_H and δ_{1H}.

A. The Arch Rib

102. General Formula for H.—Recalling the theory of the deflection of curved beams (see Chapter I, page 32), we may write for the horizontal deflection of B in the arch rib of Fig. 143

$$\delta_B = \delta'_B + H\delta_{1B} = 0 = \int_A^B \frac{Mmds}{EI} - \int_A^B \frac{Nnds}{AE} + \int_A^B \frac{Nmds}{AE\rho}, \qquad (48)$$

where

M, N = true moment and true axial thrust, respectively at any point (x, y) of the arch.

m, n = the moment and axial thrust at any point due to $H = 1$, no other forces acting.

For arches with a considerable rise, the effect of the axial thrust on the deflection is altogether negligible and for any but very flat arches it is quite small. It therefore appears permissible to assume that the thrust is approximately parallel to the arch axis, i.e., that $N = H$

* Some references will be found in the bibliography, pages 358 to 363.

† We may also think of the two-hinged arch as statically equivalent to a three-hinged arch in which, in the case of the arch rib for example, an external moment pair is applied at the crown hinge sufficient to preserve a common tangent.

sec α (see Fig. 143a). The error involved in this assumption is far too small to have any important effect on the final result. We note further that $m = y$; $n = \cos\alpha$; $dx = ds \cos\alpha$. Therefore, substituting in (48),

$$\int_A^B \frac{Myds}{EI} - H\left[\int_A^B \frac{dx \sec\alpha}{AE} - \int_A^B \frac{yds \sec\alpha}{AE\rho}\right] = 0. \quad . \quad (48a)$$

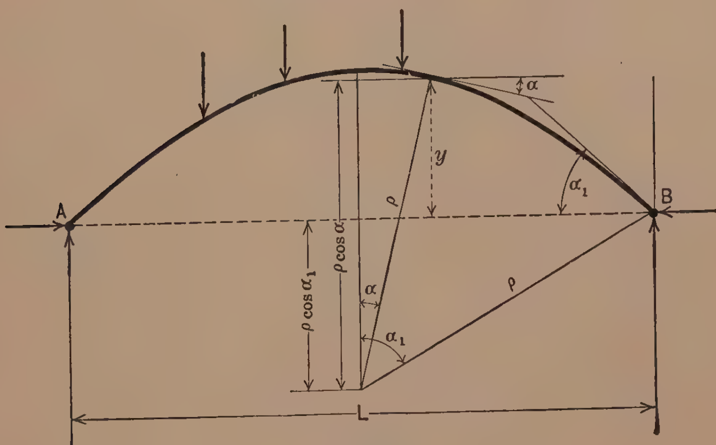


FIG. 143

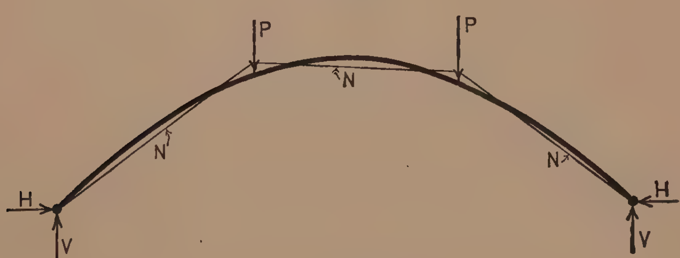


FIG. 143a

But, $M = M' - Hm = M' - Hy$, if M' = the moment at (x, y) in the structure AB acting as a simple beam. Also we may express ρ in terms of y and α thus: *

$$y = \rho\cos\alpha - \rho\cos\alpha_1, \quad \text{(Fig. 143)},$$

whence

$$\int_A^B \frac{dx \sec\alpha}{AE} - \int_A^B \frac{yds \sec\alpha}{AE\rho} = \int_A^B \frac{dx \sec\alpha}{AE} + \int_A^B \frac{ds \sec\alpha \cos\alpha_1}{AE} - \int_A^B \frac{ds}{AE}$$

$$= \int_A^B \frac{ds \sec\alpha \cos\alpha_1}{AE}.$$

* See Johnson, Bryan and Turneaure, "Modern Framed Structures," Part II, page 136. It is assumed that the curvature is approximately uniform.

We then have from (48a)

$$\int_A^B \frac{M'yds}{EI} - H\left[\int_A^B \frac{y^2ds}{EI} + \int_A^B \frac{ds \sec \alpha \cos \alpha_1}{AE}\right] = 0,$$

and

$$H = \frac{\displaystyle\int_A^B \frac{M'yds}{EI}}{\displaystyle\int_A^B \frac{y^2ds}{AE} + \int_A^B \frac{ds}{EA \cos \alpha} \cdot \cos \alpha_1}. \quad . \quad . \quad . \quad (49)$$

The second term in the denominator represents the effect of axial distortion (rib-shortening) on the value of H. For all except *very* flat arches it is so small * that it may be safely neglected. Where it is desirable to take account of the term, it will ordinarily be quite accurate enough to assume that A varies as $\sec \alpha$,—(even if this is only very roughly approximate)—whence, if A_c = area at crown,

$$A_c = A \cos \alpha,$$

and

$$\int_A^B \frac{ds}{EA \cos \alpha} \cdot \cos \alpha_1 = \frac{L_a \cos \alpha_1}{EA_c},$$

if L_a = length of arch axis.

We then have

$$H = \frac{\displaystyle\int_A^B \frac{M'yds}{EI}}{\displaystyle\int_A^B \frac{y^2ds}{EI} + \frac{L_a \cos \alpha_1}{EA_c}}. \quad . \quad . \quad . \quad . \quad . \quad (49a)$$

It is evident that the right hand term of equation (49a) is equal to $-\frac{\delta'}{\delta_1}$. For reasons just stated we shall assume in the remainder of the treatment of the arch rib that the second term of the denominator may be neglected and that

$$H = \frac{\displaystyle\int_A^B \frac{M'yds}{EI}}{\displaystyle\int_A^B \frac{y^2ds}{EI}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

* Johnson, Bryan and Turneaure, "Modern Framed Structures," Part II, page 138, estimate the error at about 1.5 per cent for a parabolic arch with a rise = $\frac{1}{5}$ of span, and a depth of rib = $\frac{1}{5}$ the rise. Kirchoff, "Statik der Bauwerke," Part II, estimates the error for a rise = $\frac{1}{7}$ to $\frac{1}{9}$ of the span, and a depth of rib = $\frac{1}{5}$ the rise at not much more than 2 per cent.

103. The Parabolic Arch with Variable Moment of Inertia.—In problem (c), Chapter II, we developed the equation for the horizontal thrust in an arch with a parabolic axis and with moment of inertia varying as sec α as

$$H = \frac{5}{8}\frac{PL}{h}(k - 2k^3 + k^4), \quad \ldots \quad (51)$$

if P is a load distant kl from the support. It is found that this equation will give fairly close results, even for a rib whose axis is not parabolic and where the variation of I departs rather widely from that assumed above. Most arch ribs arising in practice can be so analyzed. Indeed it may be used as a rough approximation for almost any two-hinged arch.

104. *Influence Lines—Moment.*—Equation (51) plotted gives the influence line for H. Remembering that $M = M' - Hy$, we may at once construct the influence line for the moment at any section by combining the simple beam moment influence line with the H influence line multiplied by the constant y. But since it is much easier to construct the simple beam influence lines than the H influence line, and since the former vary with the sections where the moment is desired while the latter is drawn once for all, it will be much more convenient to write

$$\frac{M_q}{y_q} = \frac{M'_q}{y_q} - H.$$

In Fig. 144 we construct $\frac{M'_q}{y_q}$ by dividing the simple beam moment at q by y_q and constructing the ordinary triangular influence line. Combining this with the H influence line, we get the influence curve for $\frac{M_q}{y_q}$ (the shaded area in Fig. 144*d*). It is a simple matter to multiply the ordinates in this diagram by the y corresponding to any section and thus get the true arch moment.

*** 105.** *Influence Lines—Shear and Thrust.*—These quantities are less important than the moments, but when desired the influence lines may be obtained in a similar manner.

For the shear normal to the arch axis, we have

$$\begin{aligned} \text{Shear} &= V_1 \cos \alpha - H \sin \alpha \text{—for unit load to the right of the section} \\ &= V_2 \cos \alpha - H \sin \alpha \text{—for unit load to left of section.} \end{aligned}$$

Since

$$V_1 = (1 - k) \times 1\# \quad \text{and} \quad V_2 = k \times 1\#,$$

* The treatment in this article follows closely that of Johnson, Bryan and Turneaure, "Modern Framed Structures," Part II, pages 153–154. See also Kirchoff, "Statik der Bauwerke," Part II, pages 209–216.

$$\text{Shear} = (1 - k) \cos \alpha - H \sin \alpha = \sin \alpha \,[(1 - k) \cot \alpha - H] \quad \ldots \text{load to right}$$

$$= \sin \alpha \,[k \cot \alpha - H] \quad \ldots \text{load to left.}$$

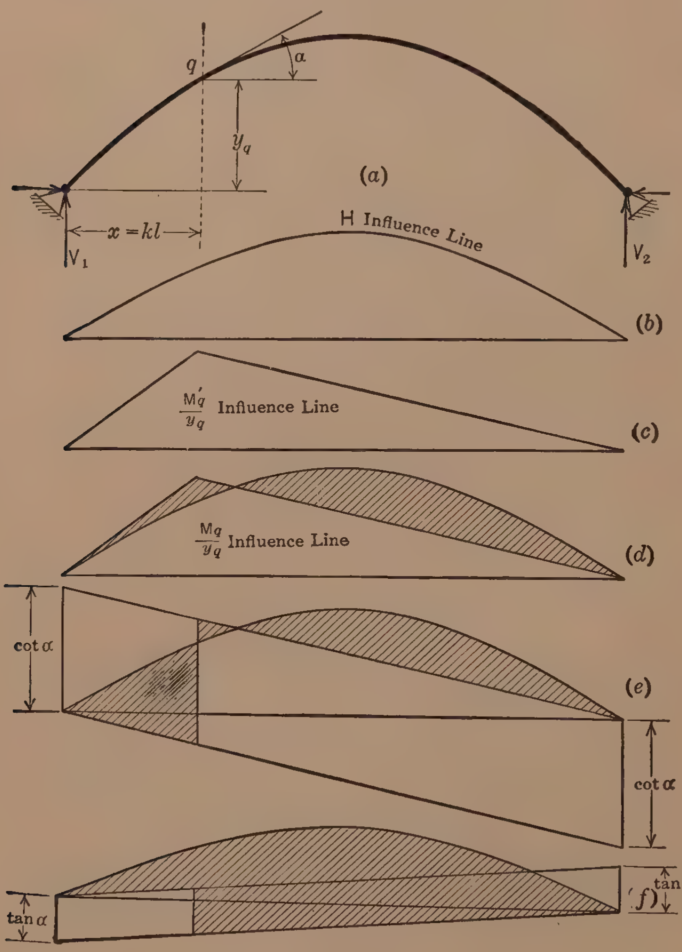


FIG. 144

Following the general method indicated for moments, it is obvious that the ordinates to the shaded diagram of Fig. 144*e* will, if multiplied by $\sin \alpha_q$ give the shear at any section q.

Since the axial thrust $N = H \cos \alpha - V_1 \sin \alpha$, or $H \cos \alpha - V_2 \sin \alpha$ according as the load is to the right or left of the section, it is

evident that in a manner similar to the case for shear we may write

$$\frac{N}{\cos\alpha} = \begin{cases} H + (1 - k)\tan\alpha \ldots \text{load right} \\ H - k\tan\alpha \ldots \text{load left.} \end{cases}$$

Fig. 144 *f* shows the influence line.

105a. *Influence Lines for Maximum Fiber Stress.*—We may obtain from the influence lines of the preceding article the moment, shear and thrust at any section due to any given loading. The thrust is a maximum under full loading for all sections, while the moment ordinarily is not, and since it is the combined effect of these two quantities which usually governs the design, it is evident that the independent influence lines do not directly give the loading producing the maximum combined stress at any section. For designing purposes it is often desirable to construct influence lines for maximum total fiber stress rather than for maximum moment and thrust. We may do this in the following manner:

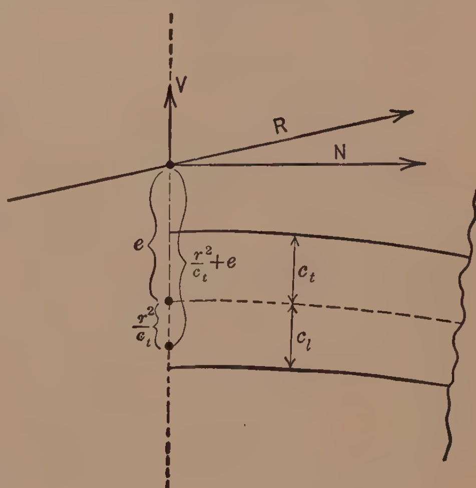


FIG. 144*b*

Let Fig. 144*b* represent any section of the arch ring. *R* is the total resultant force at the section and we may resolve as shown into the shear *V* and normal thrust *N*. Then the moment must equal *Ne* if e = arm of *N* referred to the neutral axis of the section. We must have for the stress in upper extreme fiber:

$$s_t = \frac{N}{A} + \frac{Mc_t}{I} = N\left(\frac{r^2 + ec_t}{I}\right) = N\left(e + \frac{r^2}{c_t}\right)\frac{c_t}{I} = M_k\frac{c_t}{I},$$

where c_t is the distance from neutral axis to the upper fiber and A, r and I have their usual significance. Since e, r and c_t are distances, evidently $e + \frac{r^2}{c_t}$ is a distance, as indicated in the figure, and $N\left(e + \frac{r^2}{c_t}\right)$ is a moment which we may call M_k = moment about the "kern point"

of the section. The kern point for the upper fiber lies $\frac{r^2}{c_t}$ below the neutral plane and the corresponding point for the lower fiber lies $\frac{r^2}{c_l}$ *above* the neutral plane, if c_l is the distance from neutral plane to lower extreme fiber.

If instead of the influence line for the moment about the neutral axis we draw for each section the influence lines for the moments about

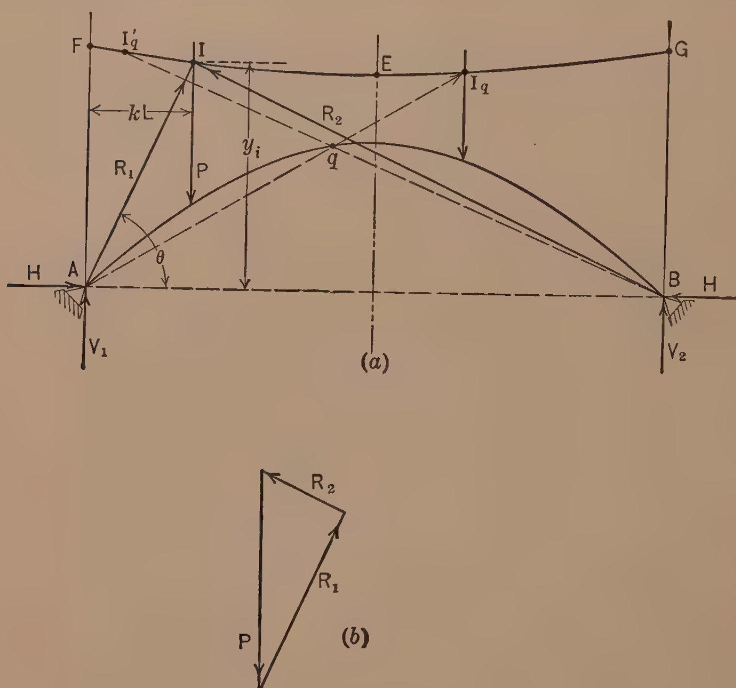


FIG. 145

the upper and lower kern points, these diagrams give us directly the loading conditions for maximum total stress on lower and upper fibers. The construction is identical with that described for the bending moment, except that in the equation $M = y\left(\frac{M'}{y} = H\right)$, y is measured to the kern point instead of to the neutral axis. Such influence lines are shown in the problem of Art. 107.

106. *Reaction Locus.*—The effect of a single moving load may be conveniently studied in a somewhat different manner. The resultant reactions R_1 and R_2 must of course always intersect on the line of action

of the load P; as this load moves across the span this point of intersection describes a locus whose equation may be readily deduced. In Fig. 145 let (kL, y_i) be the coordinates of the reaction intersection, I. Since V_1 and H are the vertical and horizontal components of R_1, we have

$H = R_1 \cos \Theta$, $V_1 = R_1 \sin \Theta$, and $\dfrac{V_1}{H} = \tan \Theta = \dfrac{y_i}{kL}$, whence from eq. (51)

$$y_i = \frac{V_1 kL}{H} = \frac{P(1-k)kL}{H} = \frac{P(1-k)kL}{\frac{5}{8}\frac{PL}{h}(k - 2k^3 + k^4)} = \frac{1.6h}{1 + k - k^2}. \qquad (52)$$

The curve of intersections is shown as FEG in Fig. 145. Once the reaction locus is constructed the two-hinged arch becomes for practical purposes statically determined, since the magnitudes of R_1 and R_2 may be determined from a simple force polygon (see Fig. 145b).

If we investigate the loading for maximum moment at q, it is clear from the figure that any load to the right of I_q or to the left of I'_q will cause negative moment; loading in the segment $I'_q - I_q$ will cause positive moment.

The exact expression for H for a partial uniform load extending from k_1L to k_2L is

$$H = \frac{5}{8}\frac{L}{h}\cdot w\int_{k_1L}^{k_2L}(k - 2k^3 + k^4)d(kL) = \frac{5}{8}\frac{wL^2}{h}\int_{k_1}^{k}(k - 2k^3 + k^4)dk$$
$$= \frac{wL^2}{16h}\Big[5(k^2 - k^4) + 2k^5\Big]_{k_1}^{k_2}.$$

107. Example.

Design of a Two-Hinged Steel Arch Rib

Span, 240 ft., Rise, 35 ft. Parabolic axis. (Fig. 146a.)
Dead load, 1800 lbs. per ft., Live load, 3200 lbs. per ft.
Impact, 25 per cent of live load.

A depth of 60 ins. will be assumed, and the rib will be designed as a box girder (Fig. 146b.)

The kern points will be assumed .8 of the half-depth distant from the center of the section for the crown section, and this factor will be assumed as .85 for the first section away from the crown and .9 for the remainder of the sections. x' and y' are coördinates of the kern points.

Section	r	$r \cdot \cos\phi$	y	y' ft.		$r \cdot \tan\phi$	x' ft.		M' ft.-lbs.		M'/y' lbs.	
	ins.	ins.	ft.	Int.	Ext.		Int.	Ext.	Int.	Ext.	Int.	Ext.
1	27	24.5	12.6	10.6	14.6	1.1	25.1	22.9	22.6	20.6	2.13	1.41
2	27	25.5	22.4	20.3	24.5	.8	48.8	47.2	39.0	37.7	1.92	1.54
3	27	26.3	29.4	27.2	31.6	.5	72.5	71.5	51.7	50.1	1.90	1.58
4	25.5	25.3	33.6	32.5	35.7	.3	96.3	95.7	57.8	57.5	1.84	1.61
5	24	24	35.0	33.0	37.0	.0	120	120	60.0	60.0	1.82	1.62

Values of H were obtained by substituting in the formula

$$H = \frac{5}{8}P\frac{l}{h}(k - 2k^3 + k^4)\ (P = 1).$$

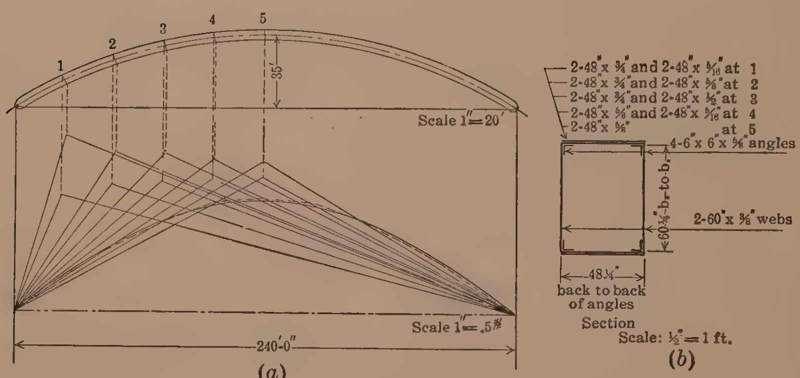


FIG. 146

Values of H and $\frac{M'}{y'}$ were plotted (Fig. 146a) and the areas of the influence diagram included between the H and the $\frac{M'}{y'}$ lines, were determined by means of a planimeter. These areas must be multiplied by y' to get the true M-areas.

108. General Method of Solution for Any Arch Rib by Means of Elastic Weights.—The majority of two-hinged arch ribs occurring in American practice are solid or open-web steel girders with either a parabolic or rather flat circular axis. The variation in I may or may not closely approximate that of $\sec \alpha$; in any case, as noted under Art. 103, the above theory gives a tolerably satisfactory approximation—sufficient for designing purposes in all ordinary cases. Where a more

Section	Areas of Influence Diagrams					
	Intradosal			Extradosal		
	+	−	Net	+	−	Net
1	8.29	3.40	+4.89	3.51	7.37	−3.86
2	5.47	3.24	+2.23	3.09	5.46	−2.37
3	4.28	2.21	+2.07	2.11	3.90	−1.79
4	2.67	1.28	+1.39	1.00	2.40	−1.40
5	1.84	.72	+1.12	.63	1.90	−1.27

Section	Moment Center	D. L. Moment, 1000″ #	L. L. Moment, 1000″ #		Maximum Moment 1000″ #	Section Modulus
			+	−		
1	Top	−12,170	24,600	51,700	−63,870	4265
	Bottom	+11,200	43,200	16,320		
2	Top	−12,550	36,499	64,200	−76,750	5120
	Bottom	+ 9,790	53,250	31,600		
3	Top	−12,220	32,000	59,200	−71,420	4760
	Bottom	+12,180	56,000	28,900		
4	Top	−10,800	17,120	41,200	−52,000	3470
	Bottom	+ 9,470	40,300	19,359		
5	Top	−10,150	11,170	33,750	−43,900	2930
	Bottom	+ 7,980	29,200	11,400		

Sections Chosen			Section		
Section	Section Modulus				
	Required	Supplied			
1	4265	4270	2-60×$\frac{3}{8}$ webs 4-6×6×$\frac{5}{8}$ angles	2 cover plates, 48×$\frac{3}{4}$	2 coverplates, 48×$\frac{5}{16}$
2	5170	5170	do	do	48×$\frac{5}{8}$
3	4760	4790	do	do	48×$\frac{1}{2}$
4	3470	3530	do	48×$\frac{5}{8}$	48×$\frac{5}{16}$
5	2930	3010	do	do	

exact analysis is desired, and for markedly irregular cases where the above theory is inapplicable we may proceed as follows:

We write the general equation for horizontal thrust

$$H = \frac{\sum \frac{M'm\Delta s}{EI}}{\sum \frac{m^2\Delta s}{EI}}, \quad \ldots \ldots \ldots \ldots (a)$$

or if it be desired to include axial thrust

$$H = \frac{\sum \frac{M'm\Delta s}{EI}}{\sum \frac{m^2\Delta s}{EI} + \frac{L_a \cos \alpha}{EA_c}}, \quad \ldots \ldots (b)$$

where Δs is any small length along the arch axis, and the summation extends over the entire arch.

In either case the denominator, as regards the loading, is a constant which for any given arch need be computed but once. If we wish to study the effect of a moving vertical load unity, M' becomes the *simple beam* moment, for a span equal to that of the arch, due to a vertical unit load. We shall call this m_V. m in equations (a) and (b) is the moment at any point of the arch due to a pair of horizontal unit forces applied at the reaction points. To avoid confusion we shall call this m_H. Then the expression

$$\sum \frac{M'mds}{EI}$$

becomes

$$\sum \frac{m_V m_H \Delta s}{EI}.$$

If the vertical unit load is applied at the point q (Fig. 147), then

δ'_{Hq} = deflection horizontally at H due to unit vertical load at q

$$= \sum \frac{m_V m_H \Delta s}{EI},$$

and

δ'_{qH} = deflection vertically at q due to unit horizontal load at a reaction point (B in Fig. 147)

$$= \sum \frac{m_H m_V \Delta s}{EI}.$$

These quantities are obviously equal, which means that if M' is the moment at any section due to unity at q, the numerator of (a) (which is actually the horizontal displacement at the support due to the unit vertical load, arch acting as a simple curved beam) may be interpreted as the *numerical* equivalent of the *vertical deflection at q due to a unit horizontal force at the support.* Obviously then the vertical deflection

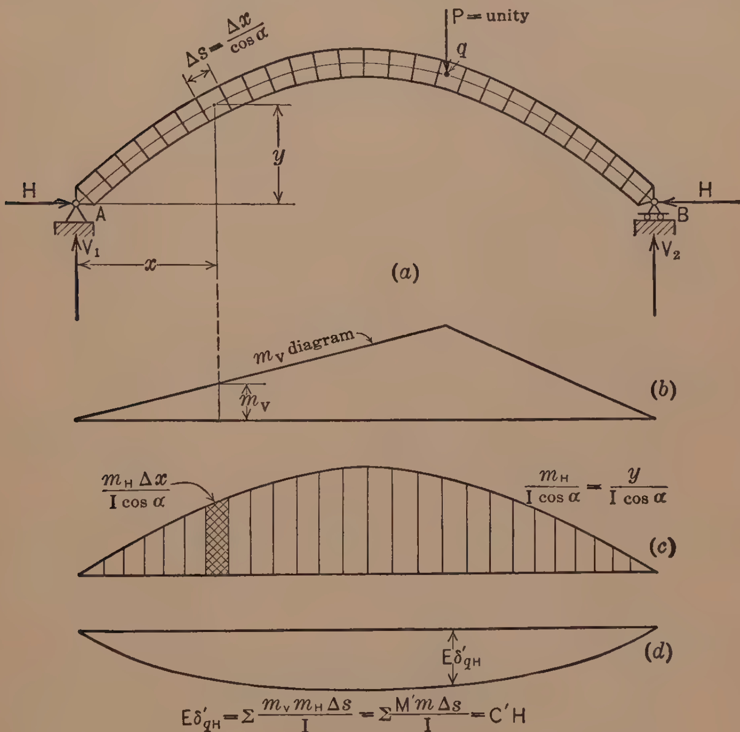


FIG. 147

diagram for all points in the arch axis due to this pair of unit horizontal forces, represents the variation of $\sum\frac{M'm\Delta s}{EI}$ as a unit load passes across the span, and is therefore to some scale, the H influence line. Calling $\sum\frac{m^2\Delta s}{I} = C$, we have, for E constant

$$H = \frac{\sum\frac{M'\Delta s}{I}m}{C} = \frac{\sum\frac{m_V m_H \Delta s}{I}}{C} = \frac{\delta'_{qH}}{C}.$$

We may apply here with slight modification the principle of elastic weights—that any simple beam deflection curve may be obtained by treating the true $\frac{\text{moment}}{EI}$ diagram as a load curve and constructing the moment diagram for this fictitious loading. We have, since $m_H = y$ and $\Delta s = \Delta x \sec \alpha$,

$$\frac{m_H \Delta s}{I} = \frac{y \Delta x}{I \cos \alpha},$$

and $E\delta'$ = moment in beam AB due to a distributed loading equal at any point to $\frac{y}{I \cos \alpha}$.

It will generally be most convenient to divide the arch axis into reasonably small segments Δs and compute $\frac{y\Delta s}{I} \doteq \frac{y \Delta x}{I \cos \alpha}$ for each one. Then assuming these quantities to act as loads (through the center point of Δs) on the simple beam AB, we may construct the moment diagram (Fig. 147*d*) either graphically by means of the string polygon, or by ordinary calculation. The ordinate to any point of this curve is equal to $\sum \frac{M'\Delta s}{I} m$ for a unit load applied to the arch at the point where the ordinate is taken, and is therefore equal to $C \times H$ for a load at this point.

This method will apply to any two-hinged arch rib and may be carried to any desired degree of accuracy by taking the segments sufficiently small. Satisfactory results will usually be obtained if ten to fifteen sections are used.

109. Example.—*Determination of true H-curve for the arch of Art. 107.* The data and results are completely shown in Table A. The values in column ⑦ were obtained by calculating the moments at tenth points in the simple beam span of 240 ft. loaded with the $\frac{y\Delta s}{I}$ values. The half division at the end was omitted; experience indicates that its effect is usually negligible.

As a typical calculation we may note that for section 2

$$R = 4.216 - \frac{1.598}{2} = 3.42.$$

$$M_2 = [R \times 2 \times 24 - 24 \times .36] \times 12 = 1866.$$

The close agreement shown in columns ⑨ and ⑩ would indicate that although the actual variation of I is quite different from that assumed, the formula for H is accurate enough for design purposes.

TABLE A

① Section	② y ft.	③ Δs ft.	④ $\frac{\Delta s}{I}$ ins.$^{-3}$	⑤ $\frac{y \cdot \Delta s}{I}$ ins.$^{-2}$	⑥ $\frac{y^2 \Delta s}{I}$ ins.$^{-1}$	⑦ $\frac{\Sigma M' y \cdot \Delta s}{I}$ lbs.-ins.$^{-1}$	⑨ H lbs.	⑩ H (Formula) lbs.	⑪ Error
1	12.6	26.4	.00238	.360	54.4	985	.411	.421	+2.5%
2	22.4	25.4	.00187	.503	135.6	1866	.778	.795	+2.2%
3	29.4	24.65	.00196	.691	244.0	2603	1.086	1.09	+0.3%
4	33.6	24.2	.00264	1.064	427.5	3141	1.312	1.274	−2.8%
5	35.0	24.1	.00380	1.598	671.0	3267	1.363	1.34	−1.6%
			$\Sigma =$	4.216	1533.5				

110. Approximate Method.
Equation (51).

$$H = \frac{5}{8}\frac{PL}{h}(k - 2k^3 + k^4),$$

gives the influence line for the horizontal thrust as a fourth degree parabola. If we replace this by a common parabola of equal area (which according to the theory of least squares should give the closest approximation) we shall have, if y_m = mid-ordinate of the equivalent parabola,

$$\tfrac{2}{3}y_m L = \frac{5}{8}\frac{L_2}{h}P\int_0^1 (k - 2k^3 + k^4)dk = \frac{PL_2}{8h},$$

and

$$y_m = \frac{3}{16}\frac{PL}{h},$$

and the approximate equation for H is

$$H = \frac{3}{4}\frac{PL}{h}(k - k^2). \qquad (53)$$

If we substitute this value in the equation for the reaction locus

$$y_i = \frac{V_1 kL}{H} = \frac{P(1-k)kL}{H},$$

we have

$$y_i = \tfrac{4}{3}h, \qquad (54)$$

i.e., the reaction locus is a horizontal straight line $\frac{4}{3}$ h above the support level (see Fig. 148).

This method furnishes an exceedingly simple solution for the parabolic arch rib with I varying as sec α. The maximum error involved is about − 4 per cent at the center and + 10 per cent at the ends. For

the central $\frac{2}{3}$ of the span (in the region where the loads are most important), the maximum error is about 5 per cent, and since the positive and negative errors tend to balance for the maximum loading at many sections, the error is still further reduced. For most arch ribs the analysis on this basis is probably as accurate as the data will justify.

111. Effects of Temperature and Yielding Supports.—From the formula $H = -\frac{\delta'}{\delta_1}$, recalling that δ' is the horizontal deflection at the support, due to any cause, in the arch acting as a simple curved beam, if α = the coefficient of expansion of the material, a change in temperature of $t°$ will cause a thrust to develop of

$$H = \frac{\pm \alpha t L}{\int_A^B \frac{m^2 ds}{EI}},$$

positive or negative, according as the temperature rises or falls.

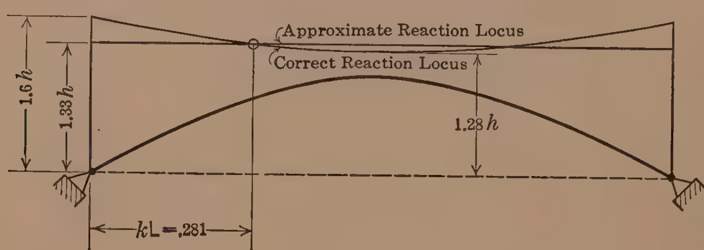


FIG. 148

If it be desired to estimate the effect of a slight horizontal yielding of the foundation, a similar method may be followed. If the yield is Δ_H we must have

$$H = -\frac{\Delta_H}{\int_A^B \frac{m_2 ds}{EI}}.$$

B. THE SPANDREL-BRACED ARCH

112. Formula for H.—The method of obtaining the horizontal thrust for a spandrel-braced arch of the type of Fig. 140 has already been indicated in Chapter II, problem (g). We have (assuming constant E)

$$H = -\frac{\delta'_H}{\delta_{1H}} = -\frac{\sum \frac{S'uL}{A}}{\sum \frac{u^2L}{A}}, \quad . \; . \; . \; . \; . \; . \quad (55)$$

where

S' = the stress in any member due to given loading, arch acting as a simply supported truss; and

u = the stress in any member due to a pair of inward horizontal unit forces acting on the same structure.

As in the arch rib we note that δ_{1H} is a constant with respect to the applied loading, hence the diagram for δ'_H must to some scale represent the H-diagram.

113. Influence Lines for H.—*First Method.*—Since the horizontal displacement at the support due to a vertical load unity at *any* point, say q, on the span, is equal to the vertical deflection at q due to a unit horizontal force at the support, it is clear that if we construct the deflection diagram for all points of load application due to this latter loading, then the ordinates to this diagram multiplied by the constant $\frac{1}{\delta_{1H}}$ will be the influence ordinates for H (see Fig. 149). This deflection diagram may be constructed by means of *a single Williot diagram* drawn for the truss loaded at each support with $H = 1$#. This will ordinarily prove the simplest method for the influence line construction.

114. Influence Line for H.—*Second Method.*—The value of the vertical deflections δ' for the horizontal unit loading may be obtained by the method of elastic weights in a manner analogous to that described in Art. 108. It was proved in Chapter I, Art. 24, that the deflection diagram of a simple truss may be represented by the moment diagram for a simple beam of the same span under suitable elastic loads. In the case of the spandrel braced arch it may be shown that the influence of the distortion of the web members on the value of H is generally negligible.* (We should note that this does not mean at all that the influence of the web members on the *deflection* is negligible; it simply means that $\frac{\delta'}{\delta_1}$ is nearly the same, whether the web members are considered or not.) The elastic loads for the chord members are the values of $\frac{\text{change of length}}{\text{moment arm}}$, and are applied vertically at the moment centers. These values of $\frac{\Delta L}{r}$ are readily computed $\left(\Delta L = \frac{uL}{AE}\right)$ and the construction of the moment diagram algebraically or graphically is then a simple matter.

* For a complete discussion and numerical comparison on this point, see H. Müller-Breslau, "Graphische Statik der Baukonstrukstionen," Band II, I Abteilung, pages 240–242. This study indicates that for all the larger ordinates to the influence line, the error is about 1 to 2 per cent.

115. Influence Lines for Truss Members.—Having determined the influence line for H, the influence line for any member of the arch truss may be found without difficulty. These influence lines may be drawn in several different ways. Remembering that

$$S = S' + Hu_H,$$

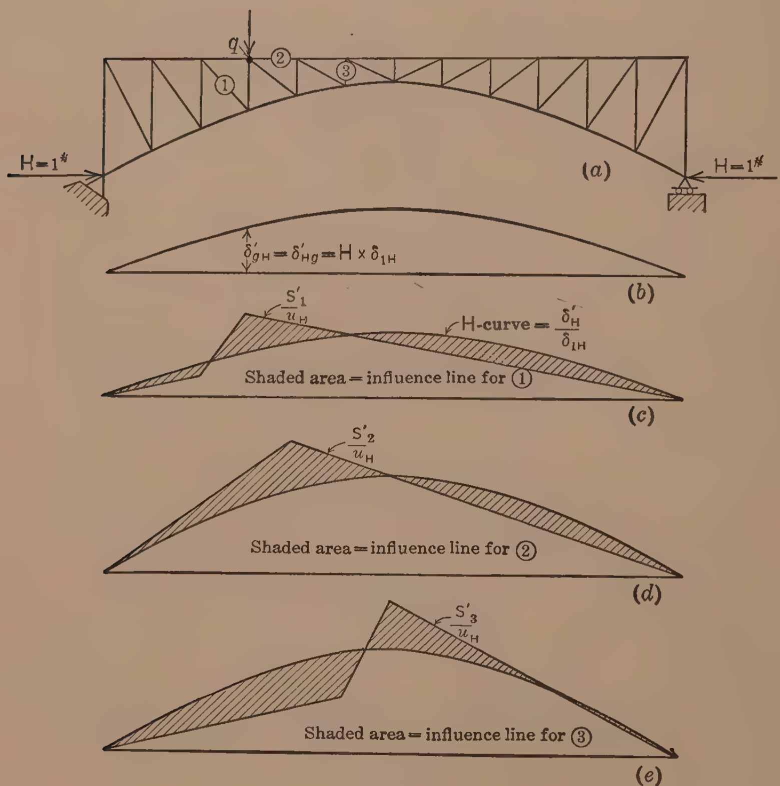


FIG. 149

where S = true stress due to a given loading in any member of arch truss

S' = stress due to a given loading in any member of arch truss when the horizontal reaction is removed,

and u_H = stress in any member due to H = unity.

We may draw the influence line for any member due to simple truss action, and correct each ordinate by $H \times u_H$. H will be obtained from the influence line and u_H from the table used for the construction of the

Williot diagram. This method is simple and results in influence lines drawn to a horizontal base. It will be somewhat more expeditious to follow a scheme similar to the one used in the analysis of the arch rib, and combine the simple truss influence line with the influence line for H. Thus

$$S = S' + Hu_H = u_H\left(\frac{S'}{u_H} + H\right)$$

and if we draw the influence line for S', dividing each ordinate by the constant u_H, we may combine this influence line directly with the H influence line, as indicated in Fig. 149*c-e*. Ordinates to the shaded curves, multiplied by u_H are the influence ordinates for the stress in the corresponding member.

116. Approximate Methods.—The formula for H (Equation 51) could not be expected to apply to a spandrel-braced arch, except as a very crude approximation. Since the formula for H

$$H = -\frac{\sum \frac{S'uL}{A}}{\sum \frac{u^2L}{A}},$$

cannot be applied until the sectional areas are known, some preliminary assumption must be made. If data are available on a somewhat similar type of structure already designed, this will greatly aid in selecting preliminary section values. These, substituted for the A's in Equation 55, will give a first approximation for H, from which a complete set of stresses and sections may be made out. If the sections so obtained differ markedly from those assumed, the calculation is repeated until substantial agreement is obtained.

In case no data such as referred to in the previous paragraph are at hand, an approximate value for H may be obtained by assuming all the sections equal, in which case the equation becomes

$$H = \frac{\Sigma S'uL}{\Sigma u^2L}$$

A still further simplification is sometimes made by assuming all the lengths equal, whence

$$H = \frac{\Sigma S'u}{\Sigma u^2}$$

It is seldom necessary to repeat the calculation more than once.

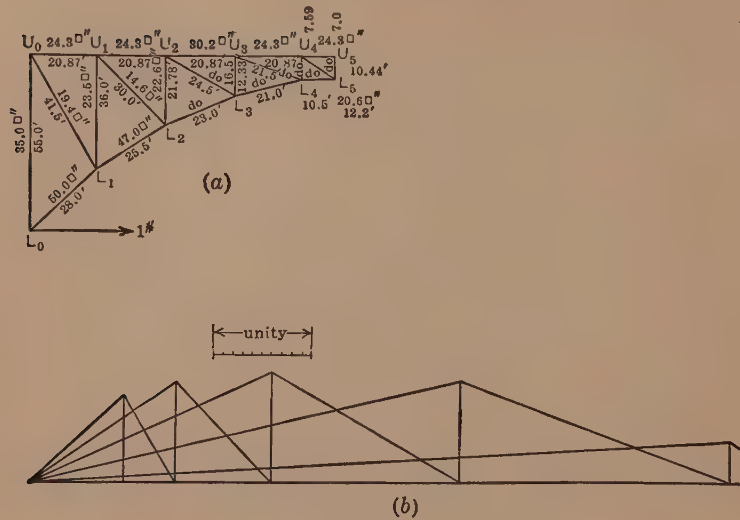


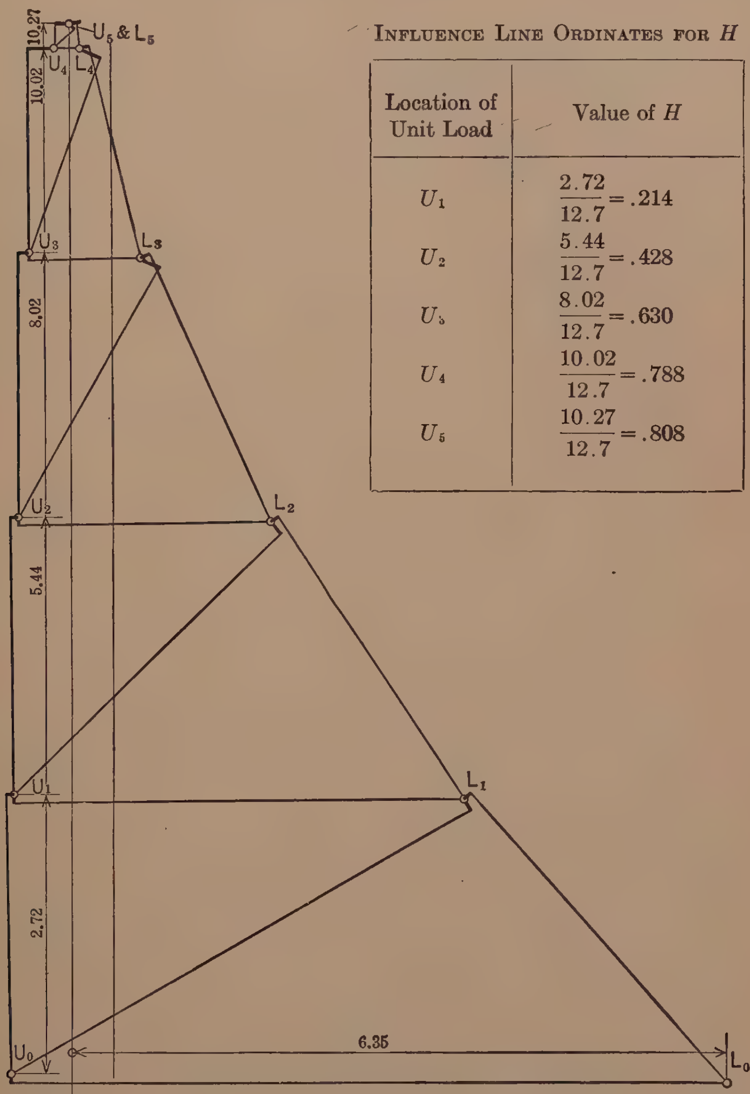
FIG. 150

TABLE A

Member	$\frac{A}{L}$	S	$\frac{SL}{A}$
U_0-U_1	.86	+ .53	+ .46
U_1-U_2	.86	+1.53	+1.31
U_2-U_3	.69	+3.46	+2.39
U_3-U_4	.86	+6.25	+5.37
U_4-U_5	.43	+6.85	+2.94
L_0-L_1	.56	−1.35	− .75
L_1-L_2	.54	−1.85	−1.00
L_2-L_3	.49	−2.77	−1.36
L_3-L_4	.45	−4.58	−2.06
L_4-L_5	.22	−7.26	−1.60
U_0-L_0	1.57	+ .89	+1.40
U_1-L_1	1.53	+1.01	+1.55
U_2-L_2	.96	+1.13	+1.09
U_3-L_3	.75	+1.03	+ .77
U_4-L_4	.46	+ .43	+ .20
U_5-L_5	.42	.0	0
U_0-L_1	2.14	−1.02	−2.18
U_1-L_2	2.06	−1.42	−2.92
U_2-L_3	1.68	−2.25	−3.78
U_3-L_4	1.47	−2.97	−4.36
U_4-L_5	.59	− .73	− .43

117. Example and Discussion.

Figs. 150 to 152, together with Tables A and B show the complete solution of a two-hinged spandrel-braced arch. The areas * and



INFLUENCE LINE ORDINATES FOR H

Location of Unit Load	Value of H
U_1	$\frac{2.72}{12.7} = .214$
U_2	$\frac{5.44}{12.7} = .428$
U_3	$\frac{8.02}{12.7} = .630$
U_4	$\frac{10.02}{12.7} = .788$
U_5	$\frac{10.27}{12.7} = .808$

FIG. 151*a*

* The arch used for illustration was originally designed as a three-hinged arch, and the areas in the table of Fig. 150 were so obtained.

lengths are given in Fig. 150a; Fig. 150b shows the stress diagram for a pair of unit horizontal loads at L_0 and L_{10}, arch acting as a simple truss. Fig. 151a shows the corresponding Williot diagram; the ordinates to the H-influence line are tabulated at the right of the figure, and the influence line is shown in Fig. 151b. Figs. 152a–e shows the construction of the influence lines from which the stresses due to live load may be obtained. It will usually be accurate enough to follow the general method illustrated in the swing bridge problem of Chapter IV, pp. 186–8, and treat the influence lines as approximate triangles for the purpose of obtaining the equivalent uniform load. For U_1U_2 for example, if AcC be taken as a triangle, we have $\frac{cc'}{Ac'} = .4$, and the proper equivalent load will be obtained from the tabular value for the .4 point in a 77 ft.

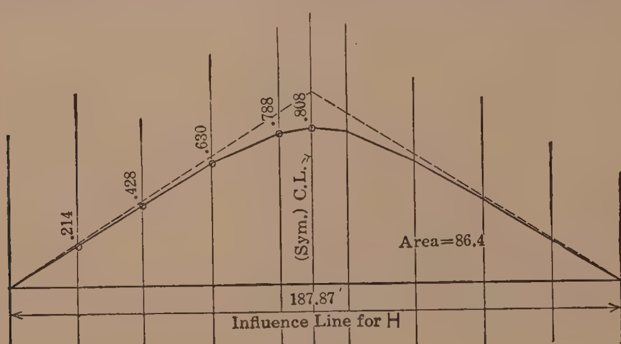


Fig. 151b

span—since C is 77 ft. from the left end. Table B gives the results for all members. The dead load stresses are taken from the calculations for the three-hinged arch, it being assumed that the structure acts thus for dead load.

118. The preceding calculation illustrates fully the method of procedure in analyzing an arch of this type. The correctness of the analysis will be indicated by a comparison of the sections designed to fit the stresses of column 13, Table B, with those originally assumed. If the discrepancy is considerable, a second calculation must be made using the revised sections. Since this calculation is identical with the preceding except that the new section areas are used, the work need not be carried further here.

119. Deflections of Two-hinged Arches.—The deflections for any arch rib may be found from the formula

$$\delta = \int_A^B \frac{Mmds}{EI},$$

where M and m are the moments in the arch due respectively to the given loads and to unit load at deflected point. But it is usually easier to solve the deflection problem by splitting it up into the deflection due

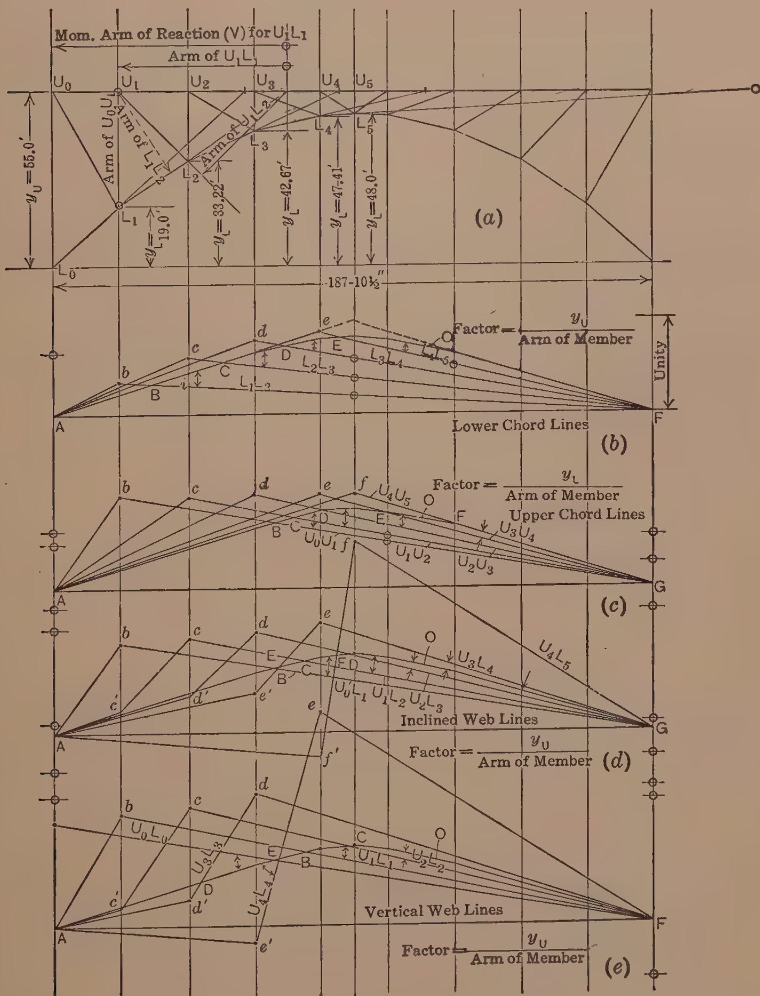


FIG. 152

to the given loads, arch rib acting at a simple beam, and the negative deflection due to H (again the arch acting as a simply supported curved beam).

TABLE B (Fig. 152)

Stress Tabulation for a Braced 2-Hinged Arch

Live Load Stresses by Influence Line Area Method (Calculations Indicated)

1	2	3	4	5	6	7	8	9	10	11	12	13
Member	+ = Tension − = Comp.	Influence Areas Loaded	Maximum Ordinate	Loaded Length	Critical Point	Area	Factor	Equivalent Uniform Live Load Cooper's E-40	Live Load Stress	Impact Stress $\% = \frac{300}{L+300}$	Dead Load Stress	Total Stress
U_0U_1	+	B–O–G	.30	117′	.3	22.1	$\frac{19}{36}$	2640	+ 31.0	+ 22.0	− 3	+ 50.0
	−	A–b–B	.78	70′	.3	27.3	$\frac{19}{36}$	2790	− 40.0	− 32.0		− 75.0
U_1U_2	+	C–O–G	.22	109′	.2	14.6	$\frac{33.2}{21.78}$	2700	+ 60.0	+ 44.0	−12	+ 92.0
	−	A–c–C	.56	77′	.4	21.6	$\frac{33.2}{21.78}$	2680	− 88.5	− 70.0		−170.5
U_2U_3	+	D–O–G	.15	102′	.2	9.0	$\frac{46.67}{12.33}$	2740	+ 85.4	+ 63.0	−33	+115.4
	−	A–d–D	.34	82′	.2	13.9	$\frac{42.67}{12.33}$	2790	−134.0	−105.0		−272.0
U_3U_4	+	E–O–G	.06	65′	.3	2.0	$\frac{47.41}{7.59}$	2880	+ 36.0	+ 30 0	−54	+ 12.0
	−	A–e–E	.19	93′	.1	8.8	$\frac{47.41}{7.59}$	2840	−156.0	−119.0		−329.0
U_4U_5	+	....	.00	....	......	....	$\frac{48.0}{7.0}$	....	....	....	− 57	
	−	A–f–F	.15	82′	.4	5.2	$\frac{48.0}{7.0}$	2650	− 94.5	− 74.0		−225.5
L_0L_1	+	....	....	....	......	....	$\frac{55.0}{41.0}$	....	....	....	−155	
	−	A–O–F	.808	188′	.5	86.4	$\frac{55.0}{41.0}$	2430	−282.0	−173.0		−610 0
L_1L_2	+	A–b–B	.123	15′	.3	1.9	$\frac{55}{29.5}$	4000	+ 14.0	+ 13.0	−133	
	−	B–O–F	.61	156′	.4	58.0	$\frac{55}{29.5}$	2500	−271.0	−178.0		−582.0
L_2L_3	+	A–c–C	.162	27′	.2	4.4	$\frac{55}{19.8}$	4000	+ 49.0	+ 45.0	−110	
	−	C–O–F	.45	133′	.3	36.0	$\frac{55}{19.8}$	2580	−258.0	−179.0		−547.0
L_3L_4	+	A–d–D	.136	37′	.2	5.0	$\frac{55}{12.3}$	3500	+ 78.5	+ 70.0	−83	+ 65.5
	−	D–O–F	.28	113′	.3	20.0	$\frac{55}{12.3}$	2670	−240.0	−174.0		−497.0

L_4L_5	+	A–e–E	.055	20′	.1	1.0	$\frac{55}{7.54}$	4000	+ 29.0	+ 27.0	−59	
	−	E–O–F	.14	98′	.2	8.3	$\frac{55}{7.54}$	2750	−167.0	−126.0		−352.0
U_0L_0	+	E–O–F	.29	116′	.3	20.5	$\frac{19}{20.87}$	2650	+ 49.5	+ 36.0	−19	+ 66.5
	−	A–o–E	.77	72′	.0	27.7	$\frac{19}{20.87}$	2920	− 73.7	− 59.0		−151.7
U_1L_1	+	B–O–F	.19	107′	.3	12.2	$\frac{55}{22}$	2670	+ 34.6	+ 25.0	−37	+ 22.2
	−	A–b–B	.96	81′	.3	38.9	$\frac{55}{22}$	2730	−112.5	− 89.0		−238.6
U_2L_2	+	C–O–F	.09	93′	.3	5.0	$\frac{55}{47}$	2680	+ 15.7	+ 12.0	−40	
	−	c'–c–C	.59	75′	.3	22.2	$\frac{55}{47}$	2770	− 72.0	− 58.0		−170.0
U_3L_3	+	A–d'–D	.14	47′	.1	3.3	$\frac{55}{53}$	3330	+ 11.4	+ 10.0	−35	
	−	D–d–F	.77	73′	.2	28.2	$\frac{55}{53}$	2830	− 83.1	− 67.0		−185.1
U_4L_4	+	A–e'–E	.80	72′	.1	28.8	$\frac{55}{135}$	2880	+ 34.0	+ 27.0	−22	+ 39.0
	−	E–e–F	1.43	116′	.1	83.0	$\frac{55}{135}$	2750	− 93.6	− 67.0		−182.6
U_0L_1	+	A–b–B	.77	69′	.3	26.6	$\frac{55}{52}$	2800	+ 78.8	+ 64.0	+ 6	+148.8
	−	B–O–G	.29	119′	.2	21.9	$\frac{55}{52}$	2670	− 62.0	− 44.0		−100.0
U_1L_2	+	c'–c–C	.59	63′	.4	18.6	$\frac{55}{38}$	2880	+ 77.6	+ 64.0	+14	+155.6
	−	C–O–G	.20	108′	.2	13.2	$\frac{55}{38}$	2720	− 52.1	− 38.0		− 76.1
U_2L_3	+	d'–d–D	.45	50′	.4	11.3	$\frac{55}{24}$	3060	+ 79.2	+ 68.0	+24	+171.2
	−	D–O–G	.10	94′	.3	5.7	$\frac{55}{24}$	2680	− 35.0	− 27.0		− 38.0
U_3L_4	+	E–e–G	.39	63′	.2	12.3	$\frac{55}{19}$	2920	+104.0	+ 86.0	+23	+213.0
	−	A–e'–E	.20	70′	.1	7.0	$\frac{55}{19}$	2920	− 59.3	− 48.0		− 84.3
U_4L_5	+	F–f–G	1.18	90′	.1	53.0	$\frac{55}{78}$	2840	+105.3	+ 81.0	+ 3	+189.3
	−	A–f'–f	1.03	92′	.1	47.3	$\frac{55}{78}$	2840	− 94.0	− 72.0		−163.0
U_5L_5	+	....	....	....	......	....	......	....	....	....	−14	
	−	....	1.00	20.87′	.5	10.43	1.0	4000	− 41.6	− 39.0		− 94.6

Thus

$$\delta = \delta' - \delta_H = \int^B \frac{M'm'ds}{EI} - H\int_A^B \frac{M'm_H ds}{EI}$$

$$= \int_A^B \frac{M'm'ds}{EI} - \frac{H}{E}\int_A^B \frac{yds}{I}\cdot m'. \quad . \quad (56)$$

The primes are introduced in the notation to indicate that the moments are simple beam moments. If the point whose deflection is sought is distant kl from the left support, m' equals $(l - k)\,x$ or $k\,(l - x)$ according as the section is to the left or the right of the deflection point. $\int \frac{yds}{I}\cdot m'$ is thus easily evaluated where y can be expressed as a simple function of x. We also note that $\sum_A^B y\frac{\Delta s}{I}\cdot m'$ is numerically equal to the moment at the point of deflection in a simple beam of the same span as the arch, acting under the elastic loads $y\frac{\Delta s}{I}$ applied at the center of the sections Δs.

If I varies as secant α, i.e., $I = I_c \sec\alpha$, the first term in the right hand member of (56) becomes

$$\int_A^B \frac{M'mds}{EI_c \sec\alpha} = \int_A^B \frac{M'm'dx}{EI_c}, \quad . \quad . \quad . \quad . \quad . \quad (57)$$

which is the formula for the deflection of a simple beam of span AB and moment of inertia equal to Ic.

Similarly for the arch truss,

$$\delta = \delta' - \delta_H = \sum\frac{S'u'L}{AE} - H\sum\frac{u'u_H L}{AE}, \quad . \quad . \quad . \quad (58)$$

where S' and u' are respectively the stresses in any member due to the given loading and to a unit load at point whose deflection is sought, arch acting as a simple truss. u_H is the stress in any member due to a horizontal thrust of unity at the support. $\sum\frac{u'u_H L}{AE}$ is the vertical deflection of the point where the unit load producing u' is applied, due to the horizontal force of unity, and all values of this summation are therefore obtained from a single Williot diagram, as explained in Art. 113. If this construction was used to obtain the influence line for H, these data are already known, and we need only evaluate $\sum\frac{S'u'L}{AE}$, which may be done algebraically or by means of a displacement diagram.

SECTION II.—THE HINGELESS ARCH RIB

120. General Equations.—(*a*) Unsymmetrical case. If we select for the statically undetermined base system, the curved cantilever beam AB, Fig. 153 (with the end B fixed) and to this beam apply the loads P and the undetermined reactions X_a, X_b, X_c as shown, we have for the three necessary conditions to determine these reactions that there shall be no horizontal or vertical displacement and no tangential rotation at A. From eqs. (29), Chapter II, we have at once that

$$\left.\begin{aligned} \delta_a = 0 &= \delta'_a + X_a\delta_{aa} + X_b\delta_{ab} + X_c\delta_{ac} \\ \delta_b = 0 &= \delta'_b + X_a\delta_{ba} + X_b\delta_{bb} + X_c\delta_{bc} \\ \delta_c = 0 &= \delta'_c + X_a\delta_{ca} + X_b\delta_{cb} + X_c\delta_{cc} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (59a)$$

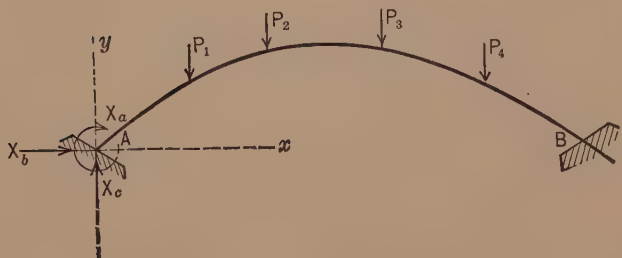


FIG. 153

These equations solve readily for the X-values. Probably evaluation by determinants is the simplest method. We have

$$X_a = -\frac{\begin{vmatrix} \delta'_a & \delta_{ab} & \delta_{ac} \\ \delta'_b & \delta_{bb} & \delta_{bc} \\ \delta'_c & \delta_{cb} & \delta_{cc} \end{vmatrix}}{\begin{vmatrix} \delta_{aa} & \delta_{ab} & \delta_{ac} \\ \delta_{ba} & \delta_{bb} & \delta_{bc} \\ \delta_{ca} & \delta_{cb} & \delta_{cc} \end{vmatrix}}$$

$$= -\frac{\delta'_a(\delta_{bb}\delta_{cc} - \delta_{bc}{}^2) + \delta'_b(\delta_{ab}\delta_{cc} - \delta_{bc}\delta_{ac}) + \delta'_c(\delta_{bc}\delta_{ba} - \delta_{bb}\delta_{ca})}{\delta_{bb}(\delta_{aa}\delta_{cc} - \delta_{ac}{}^2) + 2\delta_{ab}\delta_{bc}\delta_{ac} - (\delta^2{}_{ab}\delta_{cc} - \delta^2{}_{bc}\delta_{aa})}, \quad (59)$$

and similar equations for X_b and X_c.

Using the following notation:

M' = moment at any section of cantilever AB due to applied loading
m_a = moment at any section of cantilever AB due to $X_a = 1$
m_b = moment at any section of cantilever AB due to $X_b = 1$
m_c = moment at any section of cantilever AB due to $X_c = 1$

and applying the general deflection formulae of Chapter I, we may write out any of the nine deflection values (it will be remembered that $\delta_{ab} = \delta_{ba}$, etc.). Thus (neglecting the effect of rib shortening as is practically always done for working formulae),*

$$\delta'_a = \int_A^B \frac{M'm_a ds}{EI}; \delta_{aa} = \int_A^B \frac{m_a{}^2 ds}{EI}; \quad \delta_{ab} = \int_A^B \frac{m_a m_b ds}{EI}; \quad \delta_{ac} = \int_A^B \frac{m_a m_c ds}{EI}.$$

If we change to the ordinary notation for the statically undetermined forces,

$$X_a = M_l, \quad X_b = H_l, \quad X_c = V_l,$$

$$m_a = 1 \qquad m_b = y \qquad m_c = x$$

Then

$$\left.\begin{aligned} \delta'_a &= \int_A^B \frac{M'ds}{EI}, & \delta'_b &= \int_A^B \frac{M'yds}{EI}, & \delta'_c &= \int_A^B \frac{M'xds}{EI}, \\ \delta_{aa} &= \int_A^B \frac{ds}{EI}, & \delta_{bb} &= \int_A^B \frac{y^2ds}{EI}; & \delta_{cc} &= \int_A^B \frac{x^2ds}{EI}, \\ \delta_{ab} &= \int_A^B \frac{yds}{EI}, & \delta_{ac} &= \int_A^B \frac{xds}{EI}, & \delta_{bc} &= \int \frac{xyds}{EI}, \end{aligned}\right\} \quad . \quad . \quad (60)$$

When the equation of the arch axis, the variation of the section from point to point along the axis and the loading are known, all the above integrals are readily evaluated. Where the axis is not a regular curve, or the variation of I does not take a simple form, it is usually best to divide the axis into small finite lengths and replace the integrals by summations, thus:

$$\delta'_b = \sum_A^B \frac{M'y\Delta s}{EI}, \quad \delta_{bb} = \sum \frac{y^2\Delta s}{EI}, \quad \delta_{ac} = \sum \frac{x\Delta s}{EI}, \text{ etc.} \quad . \quad (60a)$$

This condition is usually met in the case of reinforced concrete arches —the most common type by far of the hingeless arch rib. If the arch

* The same observations apply in general to the hingeless arch as to the two-hinged arch (see page 252) on this point. For a parabolic arch in which I varies as sec α, for a rise = one-eighth of the span and crown depth = one-eighth the rise, the rib shortening effect on H would be approximately 1.7 per cent. In Turneaure and Maurer, "Principles Reinforced Concrete," pages 361–362, complete calculations are shown for an arch with a rise = one-fifth of span, crown depth = one-eighth rise, but with the axis varying considerably from a true parabola, and with I increasing very much more rapidly toward the springing line than sec α. For this case, H_c, omitting rib shortening, = 76,700#; H (due to rib shortening) = 1240#, a discrepancy of 1.6 per cent.

ring is divided into, say, twenty sections, the results will be accurate enough for all ordinary cases. In very long spans and for certain special conditions, smaller divisions may be required.

Equations (60) and (60a) substituted in the general equations of Art. 120, will suffice for the solution of any fixed ended arch. Most such arches as actually built are symmetrical, and in such case considerable simplification in the work may be effected.

(b) Symmetrical Case. We may conveniently divide the arch into two equal cantilevers by a section at the crown, and take for the statically undetermined quantities the crown shear, thrust and moment (see Fig. 154). We assume x positive to the left for the left side, to the right for the right side, and y positive downward. The deflections δ in this case are the *relative deflections* of the *cut faces* at C. We may write at once

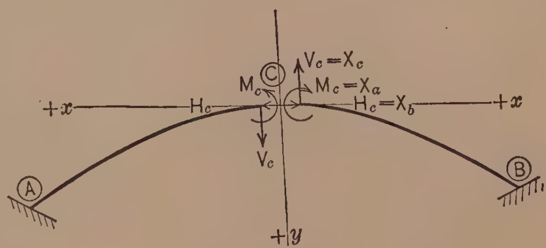


FIG. 154

$$m_a = 1 \ldots \text{for both right and left halves of arch ring,}$$

$$m_b = y \ldots \text{for both right and left halves of arch ring,}$$

$$m_c = \begin{cases} -x \ldots \text{for left half} \\ +x \ldots \text{for right half.} \end{cases}$$

We shall then have:—

$$\delta'_a = \int_C^A \frac{M'_l ds}{EI} + \int_C^B \frac{M'_r ds}{EI} = \int_A^B \frac{(M'_r + M'_l)ds}{EI} = \int_A^B \frac{M'ds}{EI},$$

$$\delta'_b = \int_C^A \frac{M'_l y ds}{EI} + \int_C^B \frac{M'_r y ds}{EI} = \int_A^B \frac{(M'_r + M'_l)ds}{EI} = \int_A^B \frac{M'yds}{EI},$$

$$\delta'_c = -\int_C^A \frac{M'_l x ds}{EI} + \int_C^B \frac{M'_r x ds}{EI},$$

$$\delta_{aa} = \int_A^B \frac{ds}{EI} = 2\int_C^A \frac{ds}{EI}; \quad \delta_{ab} = \delta_{ba} = \int_A^B \frac{yds}{EI} = 2\int_A^C \frac{yds}{EI};$$

$$\delta_{bb} = \int_A^B \frac{y^2 ds}{EI} = 2\int_A^C \frac{y^2 ds}{EI}; \quad \delta_{ac} = \delta_{ca} = -\int_A^C \frac{xds}{EI} + \int_C^B \frac{xds}{EI} = 0,$$

$$\delta_{cc} = \int_A^B \frac{x^2 ds}{EI} = 2\int_A^C \frac{x^2 ds}{EI}; \quad \delta_{bc} = \delta_{cb} = -\int_A^C \frac{xyds}{EI} + \int_C^B \frac{xyds}{EI} = 0.$$

The general equations of condition (page 277) now reduce to

$$\delta_a = 0 = \delta'_a + X_a\delta_{aa} + X_b\delta_{ba},$$

$$\delta_b = 0 = \delta'_b + X_a\delta_{ab} + X_b\delta_{bb},$$

$$\delta_c = 0 = \delta'_c + X_c\delta_{cc},$$

whence

$$X_a = -\frac{\delta'_a\delta_{bb} - \delta'_b\delta_{ab}}{\delta_{aa}\delta_{bb} - \delta^2_{ab}}; \quad X_b = -\frac{\delta'_b\delta_{aa} - \delta'_a\delta_{ab}}{\delta_{aa}\delta_{bb} - \delta^2_{ab}}; \quad X_c = -\frac{\delta'_c}{\delta_{cc}}. \quad (61)$$

Using the ordinary notation for the statically undetermined quantities $X_a = M_c$, $X_b = H_c$, $X_c = V_c$, and substituting the values for the δ's derived on page 277, we have finally (if E is constant)

$$M_c = -\frac{\int_A^B \frac{M'ds}{I}\cdot\int_A^B \frac{y^2ds}{I} - \int_A^B \frac{M'yds}{I}\cdot\int_A^B \frac{yds}{I}}{\int_A^B \frac{ds}{I}\cdot\int_A^B \frac{y^2ds}{I} - \left(\int_A^B \frac{yds}{I}\right)^2} \quad . \quad . \quad . \quad (62)$$

$$H_c = -\frac{\int_A^B \frac{M'yds}{I}\cdot\int_A^B \frac{ds}{I} - \int_A^B \frac{M'ds}{I}\int_A^B \frac{yds}{I}}{\int_A^B \frac{ds}{I}\int_A^B \frac{y^2ds}{I} - \left(\int_A^B \frac{yds}{I}\right)^2}. \quad . \quad . \quad (63)$$

$$V_c = -\frac{\int_C^B \frac{M'_r xds}{I} - \int_C^A \frac{M'_l xds}{I}}{\int_A^B \frac{x^2ds}{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

From the general equation for δ_a we get

$$X_a = -\frac{X_b\delta_{ba} + \delta'_a}{\delta_{aa}}, \quad \text{i.e.}$$

$$X_a = M_c = -\frac{H_c\int_A^B \frac{yds}{I} + \int_A^B \frac{M'ds}{I}}{\int_A^B \frac{ds}{I}}, \quad . \quad . \quad . \quad . \quad (62a)$$

a more convenient form for numercial evaluation if H_c is obtained first.

For irregular cases, or any case where y and $\frac{ds}{I}$ are not simply expressed as functions of x, we proceed as indicated on page 260, dividing each half arch into a number of finite lengths Δs, computing the average I for the portion, and substituting $\frac{\Delta s}{I}$ for $\frac{ds}{I}$, and then taking (x, y) as the coordinates of the center of the section Δs, and replacing the

integral signs by summation signs. Eqs. (62), (63) and (64) are thus readily evaluated. By taking the lengths Δs sufficiently small, the results may be obtained to any desired degree of accuracy, as noted previously, for symmetrical arches of ordinary span. All practical requirements will usually be satisfied if the half-arch is divided into 10 divisions.

Having obtained the moment, shear and thrust at the crown, we may obtain the moment at any point from the equation,

$$M = M' + M_c - H_c y \pm V_c x.$$

The moments (M, M', M_c) are considered positive when they tend to compress the outer fiber; H_c and V_c are positive when acting as indicated in Fig. 154. The plus sign before V_c applies to the right side and the minus to left side.

Note.—Since most of the applications of the hingeless arch theory which the engineer is required to make will be to reinforced concrete structures, it may be worth while to note the small modifications necessary to bring equations (62), (63), and (64) into conformity with those usually found in special treatises on the concrete arch. There is unfortunately no universally accepted standard notation, but it is believed that the form of arch equation most widely used is that given in Turneaure and Maurer's "Principles of Reinforced Concrete." (Also followed in Hool's special treatise on "Reinforced Concrete Arches," and used in Hool and Johnson's "Reinforced Concrete Engineer Handbook.")

We should first note one important simplification which is almost universally used in the standard analysis of concrete arches.*

Instead of making the divisions Δs of equal lengths or of arbitrarily varying lengths, we may adjust the divisions so that $\frac{\Delta s}{I}$ is a constant, and in such case of course, the term disappears entirely from the formulas for H_c, M_c and V_c. If we use the notation "m" for the cantilever moment instead of M', and note that if n = number of divisions in the half arch,

$$\int_A^B \frac{ds}{I} = \sum_A^B \frac{\Delta s}{I} = 2n\frac{\Delta s}{I},$$

equation (63) at once goes into

$$H_c = \frac{\sum_A^B my \cdot 2n - \sum_A^B m \cdot 2\sum_A^C y}{\left(2\sum_A^B y\right)^2 - 2n \cdot 2\Sigma y^2} = \frac{n\sum_A^B my - \sum_A^B m \cdot \sum_A^C y}{2\left[\left(\sum_A^C y\right)^2 - n\sum_A^C y^2\right]}, \qquad (65)$$

* Due apparently to Robert Schönhöfer—"Statische Untersuchungen von Bogen und Wölbtragwerken."

and similarly

$$M_c = -\frac{\sum\limits_A^B m + 2H_c \sum\limits_A^C y}{2n}. \quad (66)$$

$$V_c = -\frac{\sum\limits_A^B (m_R - m_L)x}{2\Sigma x^2}. \quad (67)$$

These are probably the most convenient forms for the equations for any case of the symmetrical fixed arch where the integral expressions in equations (62), (63) and (64) are not easily obtained.

121. Alternative Forms for the General Equations.—(*a*) If the origin of coordinates (see Fig. 155) be shifted downward to the point o, a distance c, it will be found that the equations for δ_{aa}, δ_{ab}, δ_{ac}, etc. (see page 279) remain in the same form as before. If then we take a value for c such that $\int_A^C \frac{yds}{I} = 0$, i.e., the axis of x passes through the center of gravity of the quantities $\frac{ds}{I}$ (the "elastic center"; see page 140), δ_{ab} vanishes. Since the remaining expressions for the δ's are unchanged, we shall have from eqs. (61), (62), (63), and (64)

$$X_a = M_0 = M_c + H_c \times c = -\frac{\delta'_a}{\delta_{aa}} = -\frac{\int_A^B \frac{M'ds}{I}}{\int_A^B \frac{ds}{I}}. \quad (62a)$$

$$X_b = H_0 = H_c = -\frac{\delta'_b}{\delta_{bb}} = -\frac{\int_A^B \frac{M'yds}{I}}{\int_A^B \frac{y^2ds}{I}}. \quad (63a)$$

$X_c = V_0 = V_c$ as in eq. (64).

For the irregular case, eqs. (65) and (66) become

$$M_0 = -\frac{\sum\limits_A^B m}{2n}. \quad (66a)$$

$$H_0 = \frac{\sum\limits_A^B my}{2n\Sigma y^2}. \quad (65a)$$

It will be seen that this transformation results in very simple and elegant formulae for the unknowns. The determination of the distance c is readily made. Taking the origin at the crown, and calling the y-coordinate referred to this origin y', we must have the distance to the elastic center O.

$$= y'_o = \frac{\int_A^B \frac{y'ds}{I}}{\int_A^B \frac{ds}{I}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

or in the case where a summation of finite quantities $\frac{\Delta s}{I}$ is used, and $\frac{\Delta s}{I}$ is made constant,

$$y'_o = \frac{\Sigma y'}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68a)$$

When an arch solution is to be made for but one or two load conditions (the most common practice is to investigate two cases—(1) full dead and live load and (2) dead load plus live load over half * the span), it may well be noted that the actual simplification of the work is not in proportion to the relative simplicity of formulas (65a) and (66a) compared with (65) and (66). The greater part of the tediousness of the solution lies in obtaining the various summations, Σy, Σy^2, Σx^2, Σm, etc. After these are obtained *numerically*, it is but a few minutes' work to substitute in eqs. (65), (66), and (67) and obtain the crown thrust, moment and shear. It will be observed that in locating the elastic center O and in evaluating M_0 and H_0 precisely as many *different* summation quantities are involved as appear in eqs. (65) and (66). The only advantage of the elastic center solution then lies in the simplified forms of the final equations for H and M, which we have just pointed out is of relatively small moment.

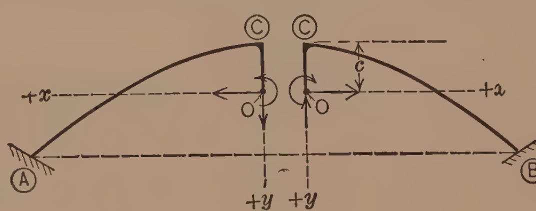


Fig. 155

Where influence lines are to be constructed, the method has other advantages which will be discussed in a later article.

(b) It has been noted many times in the preceding chapters that in

* Placing the live load over five-eighths the span is a not uncommon practice.

most cases of analysis of an indeterminate structure, more than one form of simple structure may be assumed for the base system. In the foregoing analysis of the fixed arch, we have assumed two symmetrical cantilevers; in article 120*a* (see Fig. 153) we assumed a single cantilever. If instead, we assume a simple curved beam as in Fig. 156 (this will serve, of course, for an unsymmetrical case equally well) we may write the three equations of condition in the following form:

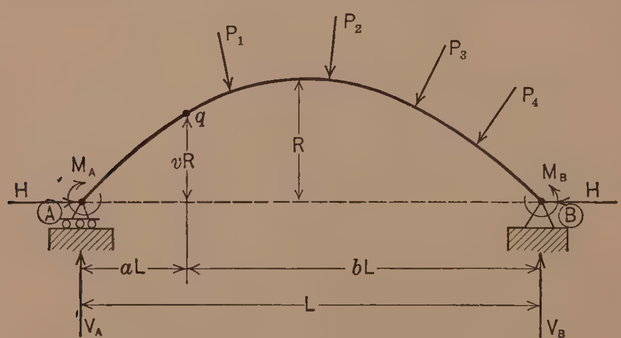


FIG. 156

(1) The deflection at A of the curved beam A-B, referred to a tangent at B must equal zero, therefore (from general equations of Chapter I)

$$\int_A^B \frac{Mxds}{EI} = 0, \quad \text{or} \quad \sum \frac{Ma\Delta s}{I} = 0, \qquad (a)$$

if E = constant.

(2) Likewise

$$\sum \frac{Mb\Delta s}{I} = 0. \qquad (b)$$

(3) Since there is no relative horizontal movement of A with respect to B,

$$\int_A^B \frac{Myds}{EI} = 0,$$

whence

$$\sum \frac{Mv\Delta s}{I} = 0. \qquad (c)$$

Now,

$$M = M' + aM_B + bM_A + HvR. \qquad (69)$$

M', M_A and M_B are taken positive when causing compression on the top (outer) fiber; H will therefore be regarded as positive when acting

outward. Substituting (69) in the fundamental formulas (a), (b) and (c), we get, since $\Sigma M_A ab = M_A \Sigma ab$, etc.

$$\Sigma M'a + M_A \Sigma ab + M_B \Sigma a^2 + HR\Sigma av = 0, \quad (70)$$

$$\Sigma M'b + M_A \Sigma b^2 + M_B \Sigma ab + HR\Sigma vb = 0, \quad (71)$$

$$\Sigma M'v + M_A \Sigma vb + M_B \Sigma av + HR\Sigma v^2 = 0. \quad (72)$$

Explicit expressions for M_A, M_B and HR are readily written out by means of determinants, thus—

$$M_A = \frac{\left\{\begin{array}{l} -\Sigma M'a\Sigma ab\Sigma v^2 + \Sigma M'b(\Sigma av)^2 - \Sigma M'v\Sigma bv\Sigma a^2 \\ +\Sigma M'v\Sigma ab\Sigma av - \Sigma M'b\Sigma a^2\Sigma b^2 + \Sigma M'a\Sigma av\Sigma bv \end{array}\right\}}{\left\{\begin{array}{c} (\Sigma ab)^2\Sigma v^2 + \Sigma b^2(\Sigma av)^2 + (\Sigma bv)^2\Sigma a^2 \\ -\Sigma a^2\Sigma b^2\Sigma v^2 - 2\Sigma ab\Sigma av\Sigma bv \end{array}\right\}} \quad (73)$$

These equations are exceedingly clumsy, and it will usually be simpler to substitute numerical values for the summations in equations (70)–(72) and solve these for the moments and thrust.

As noted, the preceding equations apply to any type of fixed arch and they are especially advantageous in certain irregular cases.* For the standard symmetrical arch, equations (65)–(67) will involve rather less detail.

It should be noted regarding this method that for symmetrical cases "v" need only be tabulated for one-half the span, while the values of "b" are the same as "a" taken in reverse order.

122. Example.—The following example will aid in making the application of preceding methods clear. We will analyze a reinforced concrete arch as shown in Fig. 157. The span is 132 ft., the rise 20 ft., thickness at crown 2 ft. and at springing line 2 ft. 6 in. Fig. 157b shows the graphical process of dividing the arch ring so that $\frac{\Delta s}{I}$ = constant. The method is as follows: Several values (usually four or five are sufficient) of I at approximately equal spaces along the arch ring are computed, and laying off $a' - u$ equal to length along one-half the arch axis, ordinates are erected at the proper points equal to the above values of I, and a smooth curve passed through them. This is approximately the correct I-diagram. Selecting a suitable number of divisions for the half arch (10 in this case), and beginning either at the crown or springing line (the latter preferably in most cases) with a trial value of Δs, a series of isosceles triangles with corresponding sides parallel are con-

* This particular form of solution was first proposed by George A. Maney. See Trans. A.S.C.E., Vol. LXXXIII, page 664 et seq.

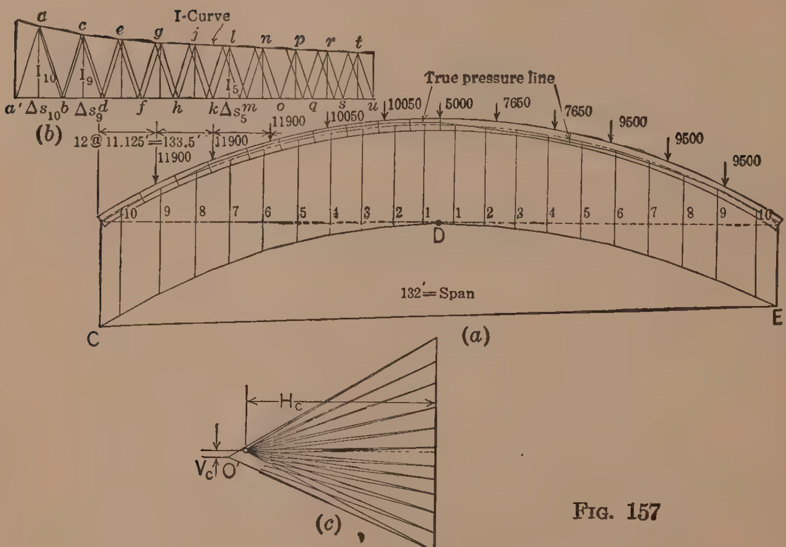


Fig. 157

TABLE A—D.L. + ½L.L.

	Dead Loads	Live Loads	Totals
I	3800#	1200	5,000
II	7650	2400	10,050
III	7650	2400	10,050
IV	9500	2400	11,900
V	9500	2400	11,900
VI	9500	2400	11,900
Total	....	....	60,800

TABLE B—D.L. + ½L.L.

Point	x	y	x^2	y^2	m_L	m_R	$(m_R+m_L)y$	$(m_R-m_L)x$
1	3.00	.05	9.0	......	−15,000	−11,500	−1,400	11,000
2	9.00	.33	81.0	.109	−45,000	−35,000	−26,000	90,000
3	15.00	.92	225.0	.85	−125,000	−90,000	−198,000	525,000
4	21.25	1.83	450.0	3.35	−210,000	−160,000	−680,000	1,060,000
5	27.58	3.21	760.0	10.30	−360,000	−270,000	−2,040,000	2,480,000
6	34.25	5.00	1170.0	25.00	−540,000	−410,000	−4,700,000	4,460,000
7	40.75	7.16	1660.0	51.27	−770,000	−595,000	−9,800,000	7,100,000
8	47.50	9.75	2250.0	95.00	−1,060,000	−820,000	−18,300,000	11,400,000
9	54.42	13.25	2955.0	175.55	−1,400,000	−1,080,000	−32,800,000	17,500,000
10	62.00	17.42	3840.0	303.56	−1,850,000	−1,430,000	−57,100,000	26,000,000
Σ		58.92	13,411.5	664.93	−11,278,000		−125,645,000	70,626,000

$$H_0=\frac{10(-125,645,000)-(-11,278,000)\times 58.92}{2\left[58.92^2-10(664.93)\right]}=93,300\#$$

$$V_0=\frac{70,626,000}{2(13,411.5)}=2635\#$$

$$M_0=-\frac{-11,278,000+2(93,300)(58.92)}{20}=\frac{+278,000}{20}=+13,900\text{ ft.-lbs.}$$

TABLE C

	m'	a	b	v	$m'a$	$m'b$	$m'v$	a^2	b^2	v^2	ab	av	bv
*L*10	225,000	.0308	.9700	.1290	7,000	218,000	29,000	.00095		.01664	.02985	.00397	
9	640,000	.0881	.9123	.3375	56,000	582,000	219,000	.00776		.11390	.08030	.02970	
8	962,000	.1405	.8600	.5125	135,000	826,000	493,000	.01960		.26260	.12075	.07200	
7	1,225,000	.1930	.8060	.6420	238,000	985,000	786,000	.03725		.41216	.15550	.12360	
6	1,440,000	.2410	.7580	.7500	347,000	1,093,000	1,081,000	.05808		.56250	.18270	.18050	
5	1,590,000	.2915	.7095	.8395	465,000	1,130,000	1,340,000	.08469		.70475	.20650	.24420	
4	1,725,000	.3400	.6610	.9085	586,000	1,141,000	1,565,000	.11560		.82500	.22500	.30820	
3	1,795,000	.3870	.6140	.9540	695,000	1,105,000	1,710,000	.14977		.91011	.23750	.36850	
2	1,840,000	.4325	.5690	.9835	795,000	1,045,000	1,809,000	.18663		.96700	.24600	.42500	
1	1,850,000	.4775	.5235	.9975	882,000	939,000	1,845,000	.22760	Σ = *do*	.99500	.25000	.47500	Σ = *do*
1	1,835,000	.5235	.4775	.9975	930,000	877,000	1,830,000	.27400				.52100	
2	1,785,000	.5690	.4325	.9835	1,015,000	774,000	1,755,000	.32376				.55900	
3	1,713,000	.6140	.3870	.9540	1,051,000	663,000	1,636,000	.37700				.58500	
4	1,619,000	.6610	.3400	.9085	1,070,000	551,000	1,472,000	.43692				.60000	
5	1,483,000	.7095	.2915	.8395	1,050,000	431,000	1,246,000	.50310				.59400	
6	1,324,000	.7580	.2410	.7500	1,000,000	319,000	995,000	.57456		Σ = *do*	Σ = *do*	.53800	
7	1,115,000	.8030	.1930	.6420	900,000	214,000	716,000	.64963				.51750	
8	870,000	.8600	.1405	.5125	748,000	122,000	446,000	.73960				.44100	
9	585,000	.9123	.0881	.3375	533,000	51,000	197,000	.83200				.30800	
*R*10	200,000	.9700	.0308	.1290	194,000	6,000	26,000	.94090				.12510	
					12,727,000	13,102,000	21,196,000	6.53940	6.53940	11.53932	3.4682	7.04927	7.04927

$$\Sigma m'a + M_B\Sigma a^2 + M_A\Sigma ab + HR\Sigma av = 0 \quad 12{,}727{,}000 + 6.5394M_B + 3.4682M_A + 7.0493HR = 0$$

$$\Sigma m'b + M_B\Sigma ab + M_A\Sigma b^2 + HR\Sigma bv = 0 \quad 13{,}104{,}000 + 3.4682M_B + 6.5394M_A + 7.0493HR = 0$$

$$\Sigma m'v + M_B\Sigma av + M_A\Sigma bv + HR\Sigma v^2 = 0 \quad 21{,}196{,}000 + 7.0493M_B + 7.0493M_A + 11.5393HR = 0$$

$$\left.\begin{aligned} 375{,}000 - 3.0712M_B + 3.0712M_A &= 0 \\ 21{,}600 - .1145M_B + 3.155M_A &= 0 \end{aligned}\right\} \begin{aligned} &66{,}900 \\ &+1.75M_A \\ &= 0. \end{aligned}$$

$$\begin{aligned} M_A &= -\ 38{,}200 \\ M_B &= +\ 83{,}700 \\ HR &= -\ 1{,}867{,}400 \\ H &= +\ 93{,}370 \end{aligned}$$

structed, as indicated in Fig. 157*b*. Each stands on a base Δs_n with an altitude I_n, and since the triangles are similar by construction, the ratio of $\frac{\Delta s}{I}$ is constant throughout. The trial assumption will not ordinarily result in an even ten divisions of the distance $a - u$, and a cut-and-try process is resorted to until this result is approximated.

The loads are assumed to be applied through spandrel columns as shown. The load values (for one-half arch) are shown in table A. Table B gives the tabular solution for the summations for the case of D.L. and $\frac{1}{2}$ L.L. The values of "*m*"—the cantilever moments—are taken from the string polygon in Fig. 157.

Arch with Fixed Ends

Solution by elastic center method

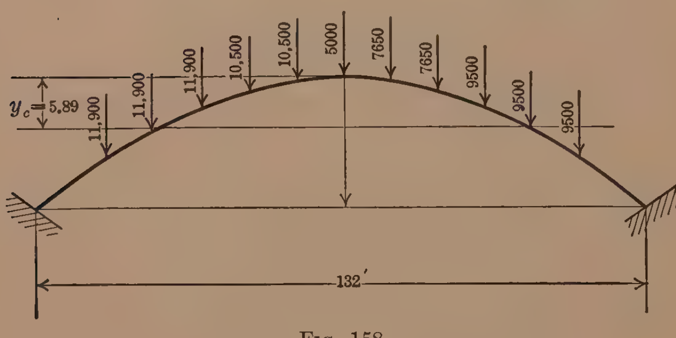


Fig. 158.

TABLE D

Point	Old "y"	New "y"
1	.05	+ 5.84
2	.33	5.56
3	.92	4.97
4	1.83	4.06
5	3.21	2.68
6	5.00	+ .89
7	7.16	− 1.27
8	9.75	− 3.86
9	13.25	− 7.36
10	17.42	−11.53

$$10\overline{)\,58.92}$$
$$5.89 = y_c$$

TABLE E

Point	x	x^2	y	y^2	M'_L	M'_R	M'_Ly	M'_Ry	M'_Lx	M'_Rx
1	3.0	9	+ 5.84	34.1	+15,000	+11,400	+87,700	+66,700	+45,000	−34,200
2	9.0	81	+ 5.56	30.8	45,000	34,200	250,000	+190,000	405,000	307,800
3	15.0	225	+ 4.97	24.6	117,000	88,200	581,500	+437,500	1,755,000	1,323,000
4	21.25	450	+ 4.06	16.4	213,875	159,163	867,000	+646,000	4,540,000	3,380,000
5	27.58	760	+ 2.68	7.2	380,580	274,328	1,020,000	+735,000	10,500,000	7,650,000
6	34.25	1170	+ .89	.8	558,875	412,600	497,000	+367,000	19,140,000	14,140,000
							+3,303,200	+2,442,200		
7	40.75	1660	− 1.27	1.6	805,225	599,500	−1,023,000	− 761,000	32,820,000	24,450,000
8	47.58	2250	− 3.86	14.9	1,102,700	825,800	4,255,000	3,185,000	52,300,000	39,200,000
9	54.42	2955	− 7.36	54.1	1,447,316	1,089,452	10,650,000	8,020,000	78,750,000	59,250,000
10	62.00	3840	−11.53	132.5	+1,908,100	+1,447,750	−22,000,000	−16,670,000	118,300,000	−89,600,000
		13,400		317.0	+6,593,671	+4,940,393	−37,928,000	−28,636,000	+318,555,000	−239,335,000
		$\Sigma x^2=26,800$		$\Sigma y^2=634.0$	+4,940,393		+3,303,200	+2,442,200	−239,335,000	

$20)\underline{11,534,064}=\Sigma M'$ $\qquad \Sigma M'_Ly=-34,624,800 \qquad -26,183,800$

$M_0=-576,703 \qquad \Sigma M'_Ry=-26,183,800 \qquad 26,800)\underline{+79,220,000}=\Sigma M'x$

$H\times y_0=5.89\times 95,900=+565,000 \qquad 634)\underline{-60,808,600}=\Sigma M'y \qquad V=+2950$

$M_c=-\ 11,700 \qquad H=+95,900$

$-M_L=1,908,000+4\times 60,800-2950\times 66-95,900\times 14.11-576,700.$

$M_L=-27,700.$

$-M_R=1,444,750+4\times 47,600+2950\times 66-95,900\times 14.11-576,700.$

$M_R=+99,000.$

The numerical values for the crown thrust, shear and moment are shown at the bottom of the table. From these, a correct reaction force polygon may be drawn as shown in Fig. 157*c*, and the true pressure line plotted (Fig. 157*a*).

122a. Table C gives the complete solutions by method (*b*) of preceding article. The string polygon *CDE* of Fig. 157*a* was used to obtain the simple beam moments M'. It is believed the table is self-explanatory.

It may be of interest to note the check between the two methods, since they are radically different in detail. We should first note that any consistent solution must give from eqs. on p. 284, $\frac{\Delta s}{I}$ = constant,

$$\Sigma M = 0 \text{ ①} \quad \Sigma M x_l = \Sigma M x_r = 0 \text{ ②} \quad \text{and} \quad \Sigma M y = 0 \text{ ③}.$$

For the first method, all three conditions were checked up. The errors were 2.6 per cent for ①, 3.5 per cent for ②, 3.0 per cent for ③.

Condition ① only was checked up for the second method, the error being about 2.5 per cent. These discrepancies are principally due to errors in scaling and small inaccuracies in computation. Considering the character of the data for the hingeless reinforced concrete arch, the check may be regarded as fairly satisfactory. Exact agreement, under such conditions, between the thrusts and moments in the two cases is hardly to be expected, though from the nature of the case, the former will agree more closely than the latter. A close check between the moment values requires extraordinary refinement in the detail calculations.

122b. As a further illustration of method, the same arch will be solved with origin at elastic center where H, M and V are applied.

The subjoined calculations show the complete solution. We first locate the elastic center, O, by the equation $y_0 = \frac{\Sigma y'}{n}$ (y' being the ordinate of any point with the crown as origin).

TABLE F

Point	M_L	M_R	Point	M_L	M_R
1	+10,500	− 3,610	6	+33,512	−22,290
2	+25,050	−17,255	7	+13,480	−21,220
3	+27,330	−32,375	8	−15,465	−19,290
4	+36,160	−34,505	9	− 4,253	+32,540
5	+20,470	−36,000	10	−42,773	+51,780

+250,882
−249,036

ΣM = +1,786

Error .36%

With the center of coordinates at the elastic center, 5.89 ft. below the crown (see Fig. 158 and Table D), the remainder of the solution is carried out in Table E.

Final moments for the various points are shown in Table F.

These check the condition $\Sigma M = 0$ to within one-third of 1 per cent. Comparison of H, M_A and M_B by the several methods is shown in table G.

TABLE G

Quantity	Method I	Method II	Method III *
H	93,300	93,370	95,900
M_A	−39,700	−38,200	−27,700
M_B	+86,300	+83,700	+99,000

123. Parabolic Arch with $I = I_c$ sec α.—Referring to equations (62), (63) and (64) and Fig. 159, we may develop general formulas for H_c, M_c and V_c just as for the two-hinged arch. The integrals entering into the equations may be evaluated thus $\left(\text{assuming } y = \frac{4h}{L^2}x^2, \text{ origin at } C\right)$.

$$\int_A^B \frac{ds}{I} = 2\int_0^{l=\frac{L}{2}} \frac{dx}{I_c} = \frac{L}{I_c};$$

$$\int_A^B \frac{yds}{I} = 2 \times \frac{4h}{L^2I_c}\int_0^{l=\frac{L}{2}} x^2dx = \frac{hL}{3I_c};$$

$$\int_A^B \frac{y^2ds}{I} = 2 \times \frac{16h^2}{L^4I_c}\int_0^{l=\frac{L}{2}} x^4dx = \frac{h^2L}{5I_c};$$

$$\int_A^B \frac{x^2ds}{I} = \frac{2}{I_c}\int_0^{l=\frac{L}{2}} x^2dx = \frac{L^2}{12I_c};$$

$$\int_A^B \frac{M'ds}{I} = \frac{P}{I_c}\int_{kl}^{l} (x - kl)dx = -\frac{PL^2}{8}(1 - k)^2;$$

$$\int_A^B \frac{M'yds}{I} = -\frac{4Ph}{L^2I_c}\int_{kl}^{l} (x - kl)x^2dx = -\frac{PhL^2}{48I_c}(3 - 8k + 16k^4);$$

$$\int_A^C \frac{M'_lxds}{I} = -\frac{P}{I_c}\int_{kl}^{l} (x - kl)xdx = -\frac{PL^3}{24I_c}(1 - 3k + 4k^3).$$

*A slight difference in the load spacing was taken in the last solution—11 ft. instead of 11.125. This would affect the final results quite appreciably. In solution (III) all moments were *computed*, in solutions (I) and (II) the moments were scaled. In each case, the results are intended to represent ordinary office practice in which no more than the required accuracy for designing purposes is sought.

We then have

$$H_c = \frac{15PL}{64h}(1 - k^2)^2; \quad \ldots \quad (74)$$

$$M_c = PL\left[\frac{(1-k)^2}{8} - \frac{5}{64}(1 - k^2)^2\right]; \quad \ldots \quad (75)$$

$$V_c = -\frac{P}{4}(2 + k)(1 - k)^2; \quad \ldots \quad (76)$$

$$= \frac{PL}{8}(1 - k)^2 - \tfrac{1}{3}H_c h;$$

$$V_B = V_c, \quad \ldots \quad (77)$$

$$V_A = \frac{P}{4}(2 - k)(1 + k)^2. \quad \ldots \quad (78)$$

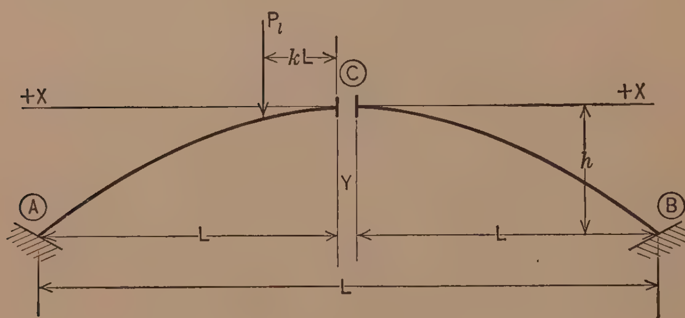


FIG. 159

From the above, we may obtain general expressions for the end moments and the moment at any point of the arch. The results are given below, the detail will be left as an exercise for the student.

$$M_A = \frac{PL}{32}(1 - k^2)(1 - 4k - 5k^2), \quad \ldots \quad (79)$$

$$M_B = \frac{PL}{32}(1 - k^2)(1 + 4k - 5k^2), \quad \ldots \quad (80)$$

$$M_x = \begin{cases} M_B + V_B\left(1 + \frac{x}{l}\right) + H_c h\left[1 - \left(\frac{x}{l}\right)^2\right]; \text{ for } x < kl. & (81a) \\ M_A + V_A\left(1 + \frac{x}{l}\right) + H_c h\left[1 - \left(\frac{x}{l}\right)^2\right]; \text{ for } x > kl\ . & (81b) \end{cases}$$

These equations give a complete solution for the hingeless arch, with parabolic axis and I varying as sec α. They apply almost exactly to a flat circular arch also, and they give a fair first approximation for almost any arch rib.

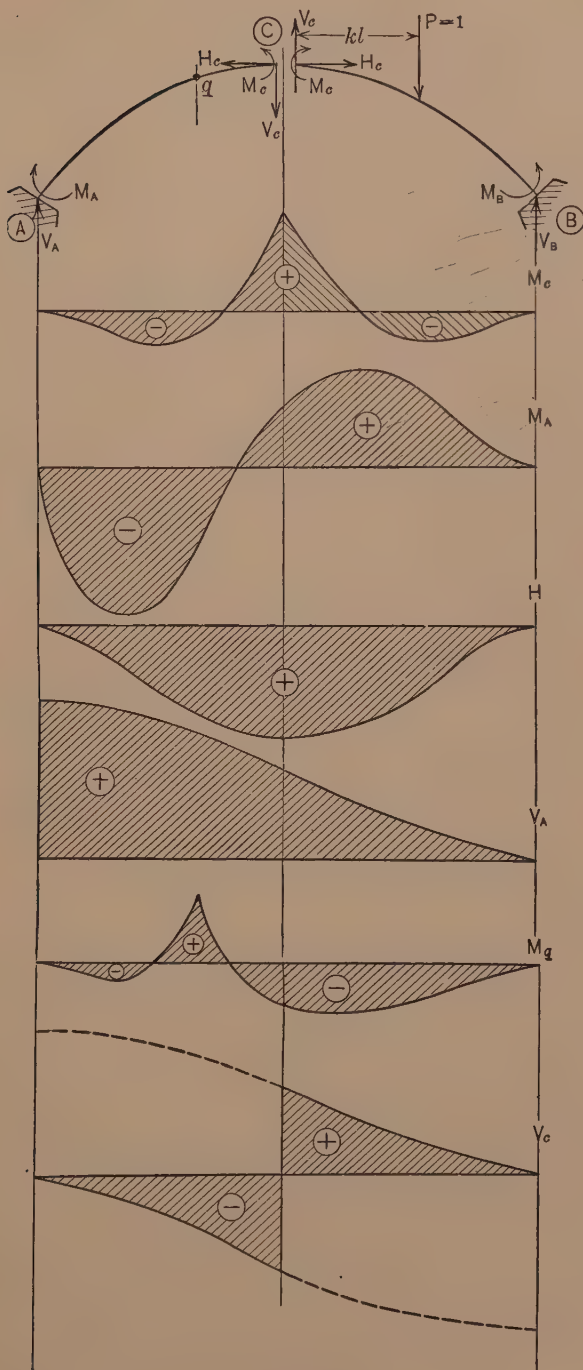


FIG. 160

124. *Influence Lines.*—Equations 74–81 when plotted for $P = 1$ # give the influence lines shown in Fig. 160. For $P = 1$# we may write the equations (74)–(78) as follows:

$$H_c = \frac{L}{h} Z_H; \qquad M_c = LZ_{M_c}; \qquad V_B = Z_{V_B}; \qquad V_A = Z_{V_A};$$

$$M_A = LZ_{M_A}; \qquad M_B = LZ_{M_B}.$$

The various values of Z for twentieth points on the span are shown in Table IX.

TABLE IX

k	Z_H	Z_{M_C}	Z_{M_A} Right	Z_{M_A} Left	Z_{V_A} Right	Z_{V_A} Left
0.0	0.469	0.0937	0.0625	0.0625	0.500	0.500
0.1	0.459	0.0493	0.0835	0.0340	0.425	0.575
0.2	0.432	0.0160	0.0960	0.0000	0.352	0.648
0.3	0.388	−0.0069	0.0995	−0.0369	0.282	0.718
0.4	0.331	−0.0203	0.0946	−0.0735	0.216	0.784
0.5	0.264	−0.0254	0.0720	−0.1055	0.156	0.844
0.6	0.192	−0.0240	0.0640	−0.1280	0.104	0.896
0.7	0.122	−0.0181	0.0430	−0.1355	0.061	0.939
0.8	0.061	−0.0102	0.0225	−0.1205	0.028	0.972
0.9	0.017	−0.0031	0.0065	−0.0790	0.007	0.993
1.0	0.0	0.0	0.0	0.0	0.0	1.000

NOTE:—Z_{M_B} and Z_{V_B} are obviously obtained by reversing columns for Z_{M_A} and Z_{V_A}. For a load on right half $V_C = V_A$; for load on left half, $V_C = V_B$.

In case of an arch for which general equations for the statically undetermined quantities are not available in such definite form, the full solution (as in Art. 122) must be carried out for a load at a number of points. For such cases we may proceed as follows: If the origin be taken at the elastic center, O (Fig. 161), we have the equation for H_0, for example (see Eq. 63a),

$$X_b = H_0 (= H_C) = -\frac{\delta'_b}{\delta_{bb}},$$

where δ'_b = the relative horizontal deflection of the faces at C due to the given applied loads. To construct the influence line for $H_0 = H_C$, we should compute this deflection for a unit load at, say, tenth points

across the span and divide the several results by the constant δ_{bb}. We recall, however, that "the horizontal deflection at the crown due to a unit vertical load at some point q, is equal to the vertical deflection at q due to a unit horizontal load at the crown" (Maxwell's principle of reciprocal deflections). If, therefore, we construct the deflection curve for the curved beam AC, loaded with a 1# horizontal load at O, we shall have, *to some scale*, the influence line for H_C. The actual value of H_C is $\frac{\delta'_{bq}}{\delta_{bb}}$. The constant δ_{bb} is readily computed, and all values of δ'_{bq} are obtained from a single deflection curve as noted above and shown in

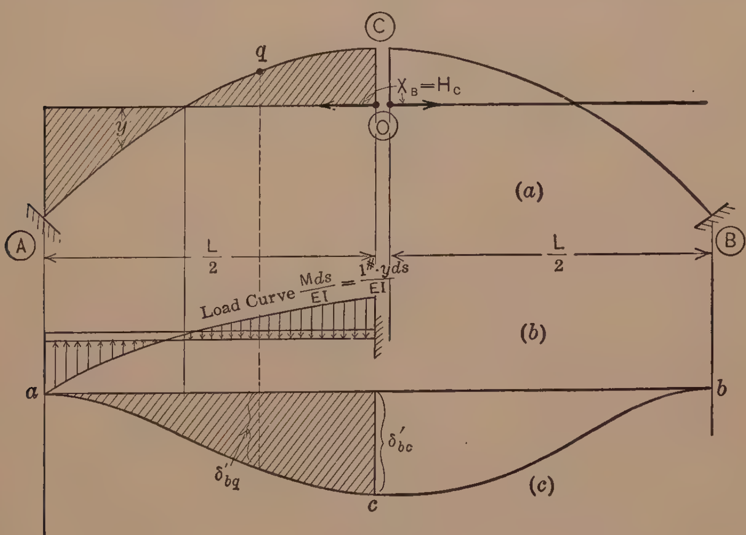


Fig. 161

Fig. 161c. This deflection curve is conveniently obtained, algebraically or graphically, as the moment curve for the straight cantilever beam * AC (Fig. 161b) under a load whose intensity at any point is $\frac{M}{EI} \times \frac{ds}{dx} = \frac{1\#y}{EI}\frac{ds}{dx}$. A similar method holds for both M_0 and V_0 ($= V_c$); that is, the influence line may be drawn as a moment diagram for the cantilever AC suitably loaded. These lines for a parabolic arch (the form would be very similar for any symmetrical arch) are shown in Fig. 162.

* See H. Müller-Breslau, "Die Graphische Statik der Baukkonstruktionen," Band II, II Abteilung, pages 560–61; also compare article by H. M. Westergaard, "Deflection of Beams by the Conjugate Beam Method,"—Journal Western Soc. of Engineers, Nov., 1921.

It is of some interest to note curves for M_A, H_A and V_A as the head of a uniformly distributed load of 1# /ft. passes across the span—the "summation influence line" (see the treatment for fixed beams, Chapter IV, page 161). Again taking the parabolic arch, the equation for

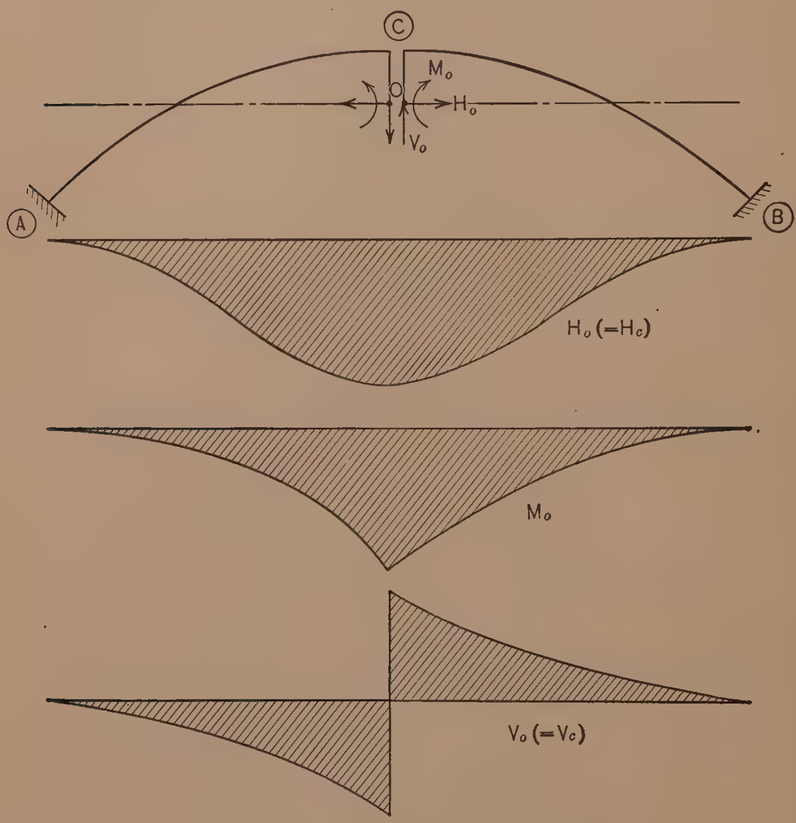


FIG. 162

$H_A = H_c$ when referred to the right support instead of the crown as the origin is (for $P = 1\#$)

$$H_A = \frac{15L}{4h}(k^2 - 2k^3 + k^4).$$

If we take $P = w \cdot d(kL) = wLdk$ and integrate the right-hand member from $k = 0$ to $k = k$, we have

$$H_A{}^W = \frac{15wL^2}{4h}\int_0^k (k^2 - 2k^3 + k^4)dk = \frac{15wL^2}{8h} \cdot k^3(10 - 15k + 6k^2). \quad (83)$$

Similarly, we have (referred to origin at right end)

$$M_A = \tfrac{2}{3}H_A h - Lk^2(1-k) = \frac{L}{2}(3k^2 - 8k^3 + 5k^4),$$

$$M_A{}^W = \frac{wL^2}{2}\int_0^k (3k^2 - 8k^3 + 5k^4)dk = \frac{wL^2}{2}k^3(1-k)^2, \quad . \quad . \quad . \quad (84)$$

and

$$V_A = k^2(3 - 2k),$$

whence

$$V_A{}^W = wL\int_0^k (3k^2 - 2k^3)dk = \frac{wLk^3}{2}(2-k). \quad . \quad . \quad . \quad . \quad . \quad (85)$$

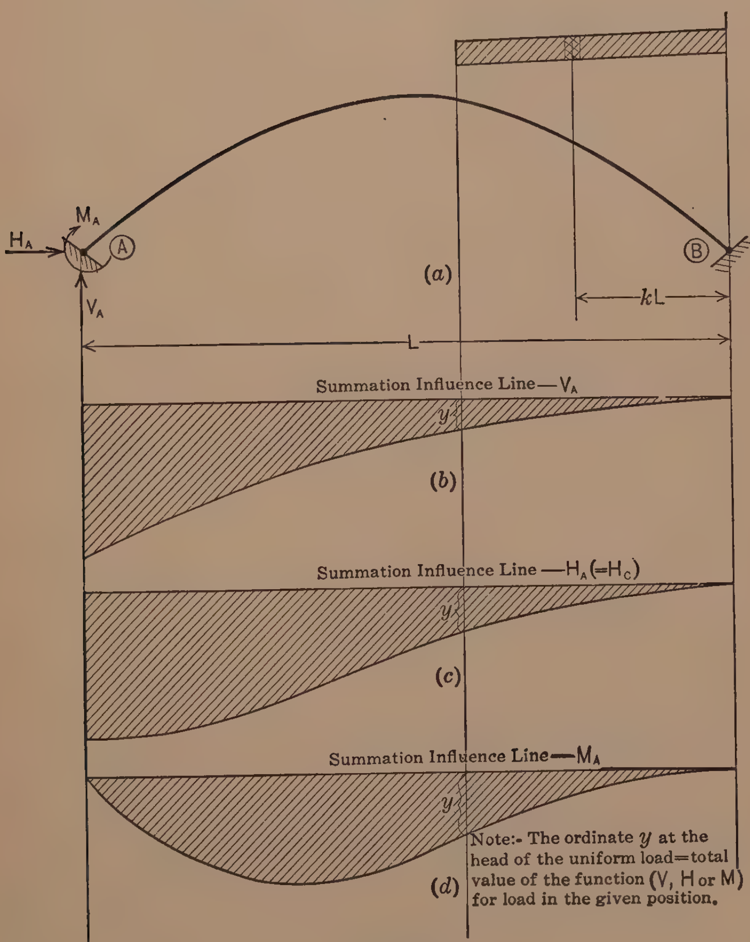


Fig. 163

Fig. 163 shows these equations plotted for $w = 1\#/\text{ft}$. By their use, the moment, shear and thrust at any point for uniform load extending from one end, partially across the span are readily obtained.

125. *Reaction Locus.*—The determination of the reactions for a hingeless arch rib for a single load anywhere on the span may be accomplished by means of the "reaction locus" method, as was done for the two-hinged arch in Art. 106. As might be expected, the construction for the hingeless arch is considerably more involved since we have not only the reaction intersection locus to determine, but also the point of application of the reactions at each support. We shall develop the method for the case of the parabolic arch with I varying as sec α.

Referring to Fig. 164, the origin is taken at the elastic center O, and the problem is to determine for any load P, distant kl from the center, the corresponding ordinates, y_k and y_q, of the intersection point of the two reactions R_l and R_r, and of the intersection of R_l with a vertical through A. Obviously, when these values are known, the direction of the reaction is fully determined and its magnitude may then be found from the force triangle, (Fig. 164*b*). We shall assume $P = 1\#$.

Taking moments about the point "k" of all forces on the left half of the arch, we shall have

$$H_0 y_k - V_0 k\frac{L}{2} - M_0 = 0,$$

whence

$$y_k = -\frac{V_0 k\frac{L}{2} - M_0}{H_0},$$

$$V_0 = V_c = -\frac{(2 + k)(1 - k)^2}{4},$$

$$H_0 = H_c = \frac{15}{64}\frac{L}{h}(1 - k^2)^2,$$

$$M_0 = M_c - H_c \cdot \frac{h}{3} = \frac{L}{8}(1 - k)^2.$$

$$\therefore\ y_k = \frac{\frac{kL}{8}(2 + k)(1 - k)^2 + \frac{L}{8}(1 - k)^2}{\frac{15}{64}\cdot\frac{L}{h}(1 - k^2)^2}$$

$$= \frac{8}{15}h\frac{(1 - k)^2(k^2 + 2k + 1)}{(1 - k^2)^2} = \frac{8}{15}h. \quad . \quad . \quad . \quad (86)$$

That is to say, the locus of reaction intersections is a horizontal line $\frac{8}{15}h$ above the axis of x (through elastic center).

We also have by inspection of Fig. 164,

$$\frac{y_k + y_q}{l - kl} = \frac{V_A}{H_A} = \frac{\dfrac{(2-k)(1+k)^2}{4}}{\dfrac{15}{64}\cdot\dfrac{L}{h}(1-k^2)^2} = \frac{8}{15}\frac{h}{l}\frac{2-k}{(1-k)^2}.$$

$$\therefore \frac{y_q}{l} = \frac{8}{15}\frac{h}{l}\frac{2-k}{1-k} - \frac{y_k}{l} = \frac{8}{15}h\frac{2-k-1+k}{l-kl} = \frac{\frac{8}{15}h}{l-kl} = \frac{y_k}{l-kl}. \quad (87)$$

This proportion affords a means to a simple and elegant graphical solution. If a unit load be placed at any point k, we draw the line

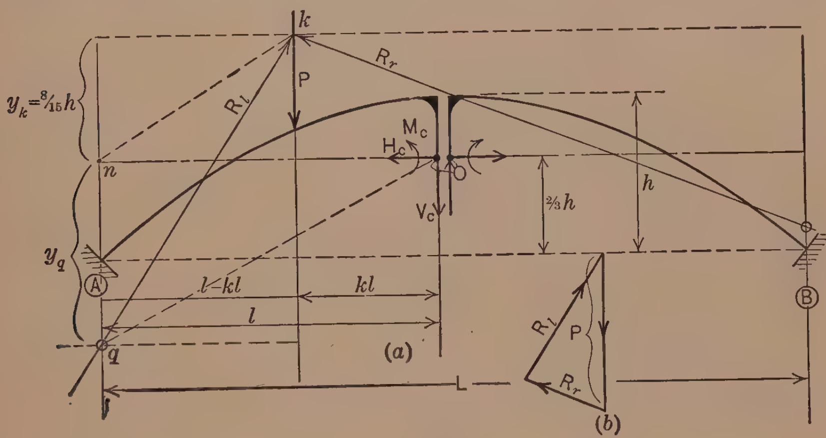


FIG. 164

kn, and from O draw a parallel line to intersection q with a vertical through A. Then $nq = y_q = y_k\dfrac{l}{l - kl}$ and qk gives the direction of R_l. A similar construction holds for R_r. Fig. 165 shows the reaction lines drawn for tenth-point loadings. It may be shown that the envelope of these lines, MON, consists of two hyperbolas having a common point of tangency with the X-axis at the origin and each a vertical asymptote through the corresponding support.

Once the reaction lines are constructed, the moment, shear, and thrust at any point are easily determined, and they also offer one of the readiest methods of determining the position of live loading for maximum positive and negative moments. Thus, if the point n (Fig. 165), is at the flange center or kern point of the corresponding section, it is clear that any loading from P_4 to P_7 will result in compression on the top fibers; any other load positions will cause tension.

The general method of reaction lines may be used in live-load investigations for any arch as an alternative method to that of influence lines, but of course, no such simple solution as that just illustrated for the parabolic arch is in general possible. If the equation for y_k is known and if the points of intersection r and s of the reaction lines with the X-axis can be determined, it is evident the reaction lines may always be drawn. The general equations are

$$y_k = \frac{V_0kl + M_0}{H_0}; \quad cl = \frac{M_0 + kl}{1 - V_0}; \quad c'l = \frac{M_0}{V}.^*$$

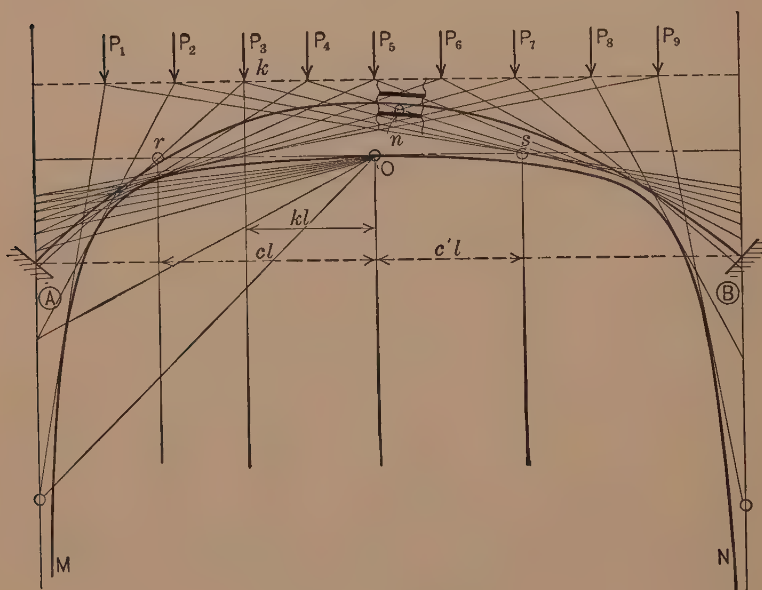


FIG. 165

Thus, as soon as M_0, H_0, and V_0 are known, the reaction line may be determined for a load at any point of the arch.

126. Effects of Temperature and Rib-Shortening.—Temperature effect is most readily seen from equations (62) and (63a), referred to the elastic center.

We have

$$H_0 = H_c = -\frac{\delta'_b}{\delta_{bb}},$$

* See H. Müller-Breslau, "Die Graphische Statik der Bau Konstruktionen," Band II, II Abteilung, page 564.

and if for δ'_b we substitute αtL, we have

$$H_0^t = H_c^t = -\frac{\alpha tL}{\int_A^B \frac{y^2 ds}{I}}. \quad \dots \dots \dots \quad (88)$$

Similarly,

$$M_0^t = -\frac{\delta'_a}{\delta_{aa}} = -\frac{\int_A^B \frac{M' ds}{EI}}{\int_A^B \frac{ds}{EI}} = -\frac{H_0 \int_A^B \frac{yds}{EI}}{\int_A^B \frac{ds}{EI}} = 0,$$

and

$$M_c^t = M_0^t - H_c^t \cdot c = \frac{\alpha tLc}{\int_A^B \frac{y^2 ds}{EI}}. \quad \dots \dots \dots \quad (89)$$

If the axis be taken at the crown, the corresponding expressions become

$$H_c^t = \frac{-\alpha tL \int_A^B \frac{ds}{EI}}{\int_A^B \frac{ds}{EI} \int_A^B \frac{y^2 ds}{EI} - \left(\int_A^B \frac{yds}{EI}\right)^2}; \quad \dots \quad (88a)$$

$$M_c^t = H_c \frac{\int_A^B \frac{yds}{EI}}{\int_A^B \frac{ds}{EI}}. \quad \dots \dots \dots \quad (89a)$$

Obvious changes are made for the case where summations are used in place of integrals.

As has been previously noted, the change in length of the axis of the rib due to direct normal stress is usually entirely negligible in its effect on H and M. If it be desired to calculate this effect it may always be done with sufficient accuracy for practical purposes by assuming the rib-shortening equivalent to a drop in temperature, and replacing, in the formula for H just given, αtL by $\frac{H_c \cos \alpha_1 \cdot L_a}{A_c E}$, the change of length due to thrust (see discussion for two-hinged arch, Art. 102, page 252).

Equation (88) then becomes

$$H_c^s = \frac{\frac{H_c \cos \alpha_1 L_a}{A_c E}}{\int_A^B \frac{y^2 ds}{EI}}, \quad \dots \dots \dots \quad (90)$$

and (88a) becomes

$$H_c^s = \frac{\dfrac{H \cos \alpha_1 L_a}{EA_c} \displaystyle\int_A^B \frac{ds}{EI}}{\displaystyle\int_A^B \frac{ds}{EI} \int_A^B \frac{y^2 ds}{EI} - \left(\int_A^B \frac{yds}{EI}\right)^2}. \quad . \quad . \quad . \quad (90a)$$

127. Deflections.—If we treat the hingeless arch as a simple curved beam acted upon by the given loads and applied end thrusts and moments equivalent to H_A, H_B, M_A and M_B, then so soon as we know these latter values, the deflection at any point of the arch may be computed by the standard beam deflection formula,

$$\delta = \int_A^B \frac{Mmds}{EI},$$

where m is the *simple beam moment*, and M = the true arch moment

$$= M' + M_c - H_c y \pm V_c x,$$

if we follow the method of Art. 120, or

$$M = M' + aM_B + bM_A + HvR,$$

if we follow the notation of Art. 121 (second method).

128. Approximate Methods.—The great majority of hingeless arch ribs met with in ordinary practice have either a parabolic axis, a flat circular axis, a compound circular axis, or an elliptical axis. Most of these types, though not all, may be fairly closely approximated by a parabolic axis. Marked differences will be found in the case of concrete arches in the variation of I, but it is tolerably well established that unless this difference be very great, the effect on the final values of the statically undetermined quantities is rather slight.* It would appear then that fair approximate results might be expected from applying the formulas for the parabolic arch with $I = I_c \sec \alpha$. Actual computations bear this out for ordinary ratios of rise to span—$\frac{1}{8}$ to $\frac{1}{4}$—and for ordinary variations of I. A fairly typical comparison is shown in table A. The arch selected is analyzed rigorously by means of influence lines in Turneaure and Maurer's "Principles of Reinforced Concrete Construction," page 362 et seq. The span is 100 ft., rise 20 ft.; the axis is not parabolic nor is the variation of I in proportion to $\sec \alpha$. [$I_c = 138$; $\sec \alpha_1 = 1.285$ for a parabolic arch of the same rise and span; therefore I at springing line should equal 1.78. The actual value is 4.33].

* For a full discussion of this point with numerical comparisons, see J. Pirlet "Statik der Baukonstruktionen," Band II, 2 Teil, pages 274–281.

TABLE A

Point	H_c		M_c		V_c	
	Actual Value	Approx. Value	Actual Value	Approx. Value	Actual Value	Approx. Value
Crown	+1.25	+1.15	+4.54	+4.60	−0.50	−0.50
$\frac{1}{5}l$	+1.14	+1.08	+0.80	+0.70	−0.36	−0.36
$\frac{2}{5}l$	+0.85	+0.85	−0.78	−1.10	−0.21	−0.21
$\frac{3}{5}l$	+0.48	+0.48	−0.92	−1.20	−0.10	−0.10
$\frac{4}{5}l$	+0.14	+0.15	−0.37	−0.50	−0.02	−0.02

The first column gives the distance from crown to the point considered (l = half span), the remaining columns are self-explanatory. There is a considerable percentage error in the relatively small moment influence line ordinates at the $\frac{2}{5}$ and $\frac{3}{5}$ points of the half span. For all other values the agreement is remarkably close.

From the preceding considerations, it seems justifiable to assume that any ordinary arch rib may be analyzed with a fair degree of approximation by substituting the parabolic arch with $I = I_c \sec \alpha$, and that for many cases, the results are so close that they may well be used in lieu of the exact values.

For more elaborate approximate methods, reference may be made to Hool and Johnson, "Concrete Engineers Handbook" (Section by Victor H. Cochrane) and to J. W. Balet, "Theory of the Elastic Arch."

129. Irregular Cases.—We shall illustrate in this article two problems of a less conventional type which are conveniently analyzed by the arch method of Art. 122*b*.

Problem I (see Fig. 166). This is a culvert section, dimensions and loading as shown in Fig. 166*a*. It may be analyzed as a fixed end curved beam AB (Fig. 166*b*). AB is not a true arch, since the length is not fixed; the tangents at A and B are fixed by the symmetry of the structure and loading. The necessary equations of condition—no angular change at A or B:

$$\Sigma aM\frac{L}{I} = 0; \quad \Sigma bM\frac{L}{I} = 0.$$

Table A and the adjoining equations show the arch solution. The M''s were obtained graphically; Fig. 166*c* shows the force polygon for given loads and the corresponding string polygon (marked "preliminary pressure line") is shown in Fig. 166*d*. When the moments M_A and M_B were obtained R_A and R_B were corrected, a new pole located on the force poly-

gon and the final pressure line drawn in Fig. 166*d*. Table B shows a comparison of the final moments as computed, and as obtained from pressure line. The average agreement is very satisfactory; and the final check $\left(\Sigma M\frac{\Delta s}{I}\right)$ is within $2\frac{1}{2}$ per cent.

PROBLEM I

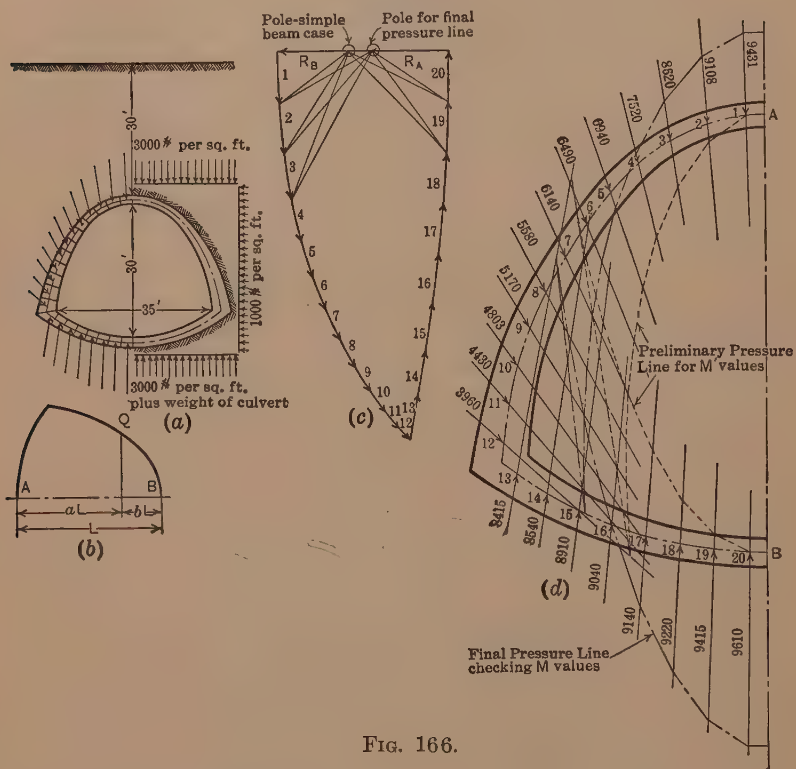


FIG. 166.

NOTE. — $\frac{L}{I}$ here signifies $\frac{\Delta s}{I}$.

Fundamental equations

$$\Sigma aM\frac{L}{I} = 0 \quad . \quad . \quad (a) \qquad\qquad \Sigma bM\frac{L}{I} = 0 \quad . \quad . \quad (b)$$

Since

$$M = M' + aM_B + bM_A,$$

(*a*) and (*b*) become

$$\Sigma ab\frac{L}{I}M_A + \Sigma a^2\frac{L}{I}M_B = -\Sigma M'a\frac{L}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a')$$

$$\Sigma b^2\frac{L}{I}M_A + \Sigma ab\frac{L}{I}M_B = -\Sigma M'b\frac{L}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b')$$

TABLE A—Problem I

	Force, Pounds	Arm Inches	M'	$\frac{L}{I}$	$M'\frac{L}{I}$	b	a	$b^2\frac{L}{I}$	$a^2\frac{L}{I}$	$ab\frac{L}{I}$	$M'b\frac{L}{I}$	$M'a\frac{L}{I}$
1	13,900	− 1	−13,900	1.000	−13,900	.995	.003	.990	.0000	.0030	−13,800	4,000
2	16,600	+ 13	+216,000	.869	+187,500	.979	.022	.830	.0004	.0185	+182,400	26,600
3	22,500	+ 28	630,000	.735	463,000	.940	.057	.649	.0024	.0397	435,000	81,100
4	29,300	+ 40	1,172,000	.641	751,000	.890	.108	.508	.0075	.0616	669,000	176,000
5	35,600	+ 49	1,743,000	.571	995,000	.822	.177	.386	.0179	.0831	817,000	291,000
6	41,500	+ 56	2,322,000	.515	1,171,000	.751	.248	.290	.0317	.0961	882,000	458,000
7	47,600	+ 68	3,240,000	.438	1,418,000	.645	.323	.182	.0458	.0913	914,000	581,000
8	52,500	+ 77	4,040,000	.360	1,450,000	.596	.401	.128	.0579	.0861	865,000	684,000
9	57,300	+ 84	4,810,000	.279	1,370,000	.515	.485	.0761	.0675	.0717	726,000	715,000
10	61,700	+ 91	5,620,000	.222	1,250,000	.428	.572	.0407	.0724	.0544	535,000	730,000
11	65,800	+ 96	6,310,000	.176	1,112,000	.342	.656	.0206	.0760	.0396	381,000	700,000
12	69,300	+101	6,930,000	.136	940,000	.252	.745	.0085	.0754	.0254	237,000	1,012,000
13	64,100	+ 97	6,220,000	.199	1,238,000	.180	.817	.0064	.1329	.0292	223,000	981,000
14	56,600	+ 86	4,870,000	.233	1,135,000	.134	.864	.0042	.1738	.0270	152,000	898,000
15	48,400	+ 72	3,480,000	.275	957,000	.096	.905	.0026	.233	.0248	94,500	699,000
16	40,400	+ 57	2,300,000	.326	850,000	.065	.932	.0014	.284	.0198	49,000	546,000
17	32,600	+ 44	1,434,000	.396	569,000	.040	.960	.0006	.365	.0087	23,000	311,000
18	25,600	+ 28	717,000	.444	318,000	.019	.979	.00016	.425	.0081	14,100	122,000
19	20,000	+ 12	240,000	.497	119,200	.006	.995	.0000	.510	.0032	7,700	4,600
20	18,000	+ 0.5	9,000	.517	4,600	.0001	.999	.0000	.516	.0002	....	
Σ	....	....	....	....	....	....	....	4.12436	3.095	0.791	7,206,700	9,020,300

Substituting summation values in (a') and (b')

$$.791M_A + 3.095M_B = -9{,}020{,}000$$
$$4.124M_A + .791M_B = -7{,}207{,}000$$

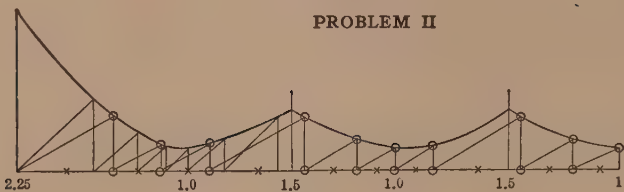
whence

$$M_A = -1{,}248{,}000 \text{ in.-lbs.}$$
$$M_B = -2{,}590{,}000 \text{ in.-lbs.}$$

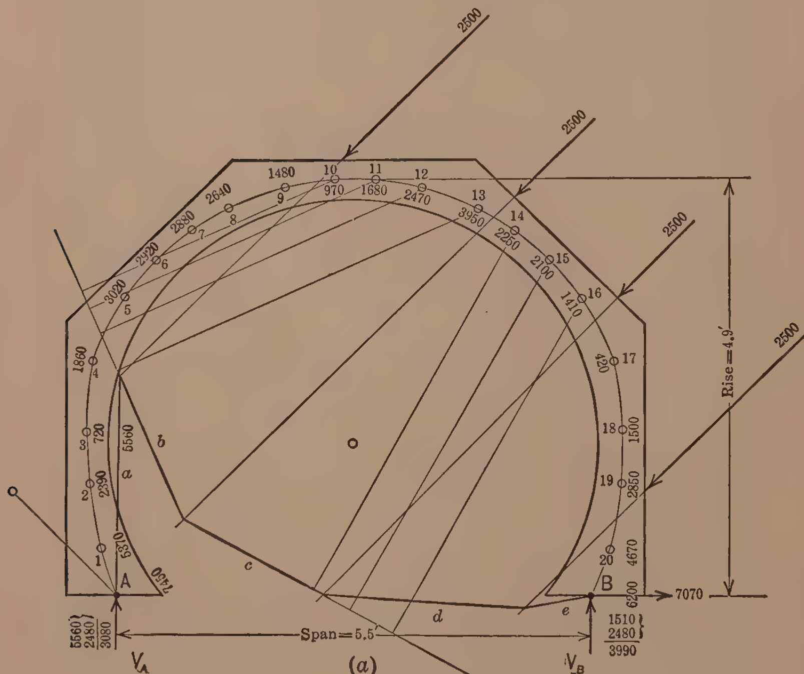
TABLE B—Problem I

Point	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Σ
M'	− 114	+ 216	+ 630	+1172	+1743	+2322	+3240	+4040	+4810	+5620	+6310	+6930	+6220	+4870	+3480	+2300	+1434	+ 717	+ 240	− 9	
aM_B	− 8	− 57	− 148	− 280	− 459	− 751	− 836	− 1038	− 1258	− 1480	− 1700	− 1930	− 2120	− 2240	− 2340	− 2410	− 2490	− 2530	− 2580	− 2590	
bM_A	− 1240	− 1220	− 1272	− 1110	− 1025	− 938	− 805	− 745	− 643	− 535	− 426	− 314	− 225	− 167	− 120	− 81	− 50	− 24	− 7	0	
M calculated	− 1362	− 1061	− 790	− 218	+ 259	+ 633	+1599	+2257	+2509	+3605	+4184	+4686	+3875	+2463	+1020	− 191	− 1106	− 1837	− 2347	− 2599	
M from pressure line	− 1248	− 1042	− 690	− 222	+ 297	+ 863	+1523	+2210	+2900	+3550	+4160	+4810	+4060	+2420	+1138	− 23	− 1000	− 1732	− 2215	− 2590	
Check $\Sigma M\frac{L}{I}=0$	− 1362	− 921	− 581	− 140	+ 148	+ 326	+ 700	+ 811	+ 700	+ 801	+ 736	+ 638	+ 770	+ 575	+ 281	− 62	− 438	− 815	− 1168	− 1340	−341 (2½%)

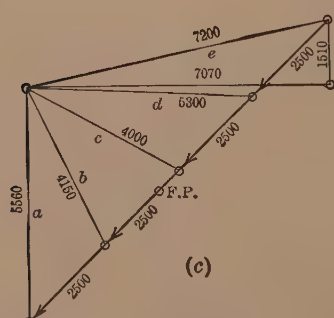
PROBLEM II



(*b*)



(*a*)



(*c*)

FIG. 167

TABLE A—PROBLEM II

No.	Thrust	e	M′	a	b	v	M′a	M′b	M′v	ab	av	bv	a²	b²	v²	M	Mv
1	5560	.15	− 830	− .03	1.03	.11	+ 20	− 850	− 90	− .031	−.003	...	.001	....	.012	+ 5,370	+ 590
2	5560	.30	− 1,670	− .05	1.05	.27	+ 80	− 1,750	− 450	− .052	−.013	...	.002	....	.073	+ 2,390	+ 650
3	5560	.30	− 1,670	− .06	1.06	.39	+ 100	− 1,770	− 650	− .064	−.023	...	.004	....	.152	+ 720	+ 280
4	5560	.25	− 1,390	− .05	1.05	.57	+ 70	− 1,460	− 790	− .052	−.028	...	.002	....	.325	− 1860	−1060
5	5560	.12	+ 670	.02	.98	.72	+ 10	+ 660	+ 480	+ .020	+.014	...	.000	....	.518	− 3,020	−2170
6	5560	.55	+ 2,060	.10	.90	.80	+ 310	+ 2750	+ 2,450	+ .090	+.080	...	.010	....	.640	− 2,920	−2330
7	5560	.90	+ 5,000	.16	.84	.88	+ 800	+ 4,200	+ 4,350	+ .134	+.139	...	.026	....	.757	− 2,880	−2510
8	5560	1.30	+ 7,230	.23	.77	.94	+ 1,660	+ 5,570	+ 6,800	+ .177	+.216	...	.053	....	.883	− 2,640	−2480
9	5560	1.94	+ 10,800	.35	.65	.99	+ 3,780	+ 7,020	+ 10,700	+ .227	+.346	...	.123	....	.980	− 1,480	−1470
10	4510	3.30	+ 14,900	.46	.54	1.00	+ 6,860	+ 8,040	+ 14,900	+ .248	+.460	...	.212	....	1.000	+ 970	+ 970
11	4510	3.70	+ 16,700	.54	.46	1.00	+ 9,010	+ 7,690	+ 16,700	+ .697	+.540	...	.291	....	5.340	+ 1,680	+1680
12	4510	4.18	+ 18,850	.65	.35	.99	+ 12,250	+ 6,600	+ 18,460	$= \frac{1}{2}\Sigma$	+.645	...	.423	....	$= \frac{1}{2}\Sigma$	+ 2,470	+2450
13	4510	4.70	+ 21,200	.77	.23	.94	+ 16,300	+ 4,900	+ 19,940	....	+.723	...	.592	....	....	+ 3,950	+3710
14	4000	4.85	+ 19,400	.84	.16	.88	+ 16,300	+ 3,100	+ 16,900	....	+.731	...	.706	....	....	+ 2,250	+1960
15	4000	4.75	+ 19,000	.90	.10	.80	+ 17,100	+ 1,900	+ 15,200	....	+.720	...	.810	....	....	+ 2,100	+1680
16	4000	4.55	+ 18,200	.98	.02	.72	+ 17,800	+ 400	+ 13,100	....	+.705	...	.960	....	....	+ 1,410	+1020
17	5300	3.00	+ 15,900	1.05	− .05	.57	+ 16,700	− 800	+ 90,50	....	+.598	...	1.110	....	....	+ 420	+ 240
18	5300	2.15	+ 11,400	1.06	− .06	.39	+ 12,100	− 700	+ 4,440	....	+.413	...	1.124	....	....	− 1,500	− 590
19	5300	1.53	+ 8,100	1.05	− .05	.27	+ 8,500	− 400	+ 2,200	....	+.283	...	1.110	....	....	+ 2,850	− 770
20	7200	.50	+ 3,600	1.03	− .03	.11	+ 3,710	− 110	+ 1,400	....	+.113	...	1.061	....	....	+ 4,670	− 510
Σ							143,500	+ 44,990	+154,090	+1.394	+6.66		8.620		+10.68	− 23,820 + 23,930	−13,890 +14,890

Attention should be called to the notation in one point—for convenience $\frac{L}{I}$ is used for $\frac{\Delta s}{I}$.

130. *Problem II* (see Fig. 167).—This problem is the analysis of a water conduit section with dimensions and loading as shown in Fig. 167*a*. It is assumed that a very heavy base along the line A–B completely fixes the structure at A and B. Fig. 167*b* shows the graphical construction to determine the division lengths rendering $\frac{\Delta s}{I}$ constant. Assuming the arch a simple beam hinged at B and on rollers at A (V_A vertical), the force polygon, Fig. 167*c*, was constructed and from it the string polygon was drawn in Fig. 167*a*. The column headed "e," Table A,

Problem II—*Solution of Equations*

$$\Sigma Ma = 0 \qquad \Sigma Mb = 0 \qquad \Sigma Mv = 0$$

$$M = M' + aM_B + bM_A + vF_HR$$

$$\Sigma M'a + M_B\Sigma a^2 + M_A\Sigma ab + F_HR\Sigma av = 0$$

$$\Sigma M'b + M_B\Sigma ab + M_A\Sigma b^2 + F_HR\Sigma bv = 0$$

$$\Sigma M'v + M_B\Sigma ab + M_B\Sigma bv + F_HR\Sigma v^2 = 0$$

TABLE B

	M_B	M_A	F_HR	Constant term
①	8.620	1.394	6.660	−143,500
②	1.394	8.620	6.660	− 45,000
③	6.660	6.660	10.680	−154,000
	1.0	.162	.772	− 16,650
	1.0	6.180	4.780	− 32,300
	1.0	1.000	1.601	− 23,130
		−6.018	− 4.008	+ 15,650
		+ .838	+ .829	− 6,480
		1.0	+ .665	− 2,600
		1.0	+ .995	− 7,740
			.330	− 5,100
			F_HR =	− 15,100
			M_A =	+ 7,450
			M_B =	− 6,200
			F_H =	− 3,080

Correction to be applied to V to obtain final value

$$= \Delta V = \frac{M_A - M_B}{L} = \frac{7450 - (-6200)}{5.5} = 2480.$$

This evidently acts downwards at A and upwards at B.

gives the arms as scaled from the string polygon, while the second column, headed "Thrust," gives the corresponding resultant force. The necessary summations are obtained from Table A. It should be recalled that vR is the vertical height of any division point measured from line A–B, and therefore v is the *relative rise* of any such point. It should also be noted that the horizontal thrust necessary to be added at A and B to secure complete arch action is denoted as F_H (instead of H as in the case of vertical loads).

TABLE C—PROBLEM II

FINAL MOMENT CALCULATIONS

Point	1	2	3	4	5	6	7
M'	− 830	−1670	−1670	−1390	+ 670	+ 3,060	+ 5,000
aM_B	+ 180	+ 310	+ 370	+ 310	− 120	− 620	− 990
bM_A	+7670	+7820	+7900	+7820	+ 7,300	+ 6,710	+ 6,250
vF_HR	−1660	−4070	−5880	−8600	−10,870	−12,070	−13,140
M	+5370	+2390	+ 720	−1860	− 3,020	− 2,920	− 2,880

Point	8	9	10	11	12	13	14
M'	+ 7,230	+10,800	+14,900	+16,700	+18,850	+21,200	+19,400
aM_B	− 1,420	− 2,170	− 2,850	− 3,340	− 4,030	− 4,770	− 5,200
bM_A	+ 5,740	+ 4,840	+ 4,020	+ 3,420	+ 2,600	+ 1,710	+ 1,190
vF_HR	−14,190	−14,950	−15,100	−15,100	−14,950	−14,190	−13,140
M	− 2,640	− 1,480	+ 970	+ 1,680	+ 2,470	+ 3,950	+ 2,750

Point	15	16	17	18	19	20
M'	+19,000	+18,200	+15,900	+11,400	+8100	+3600
aM_B	− 5,580	− 6,070	− 6,510	− 6,570	−6510	−6390
bM_A	+ 750	+ 150	− 970	− 450	− 370	− 220
vF_HR	−12,070	−10,870	− 8,600	− 5,880	−4070	−1660
M	+ 2,100	+ 1,410	+ 420	− 1,500	−2850	−4670

Table B shows the simultaneous solution for M_A, M_B, and F_H, and Table C gives the final moment calculations. The moments are tabulated on Fig. 167*a* also, on the *tension side* of the arch ring. That is, at point 7 the moment is 2880# producing tension on *outside* fiber; at point 12 the moment is 2470# producing tension on *inner* fiber. The check ($\Sigma M = 0$) is within less than $\frac{1}{2}$ of 1 per cent; ΣMv checks within 3.8 per cent.

131. Influence Lines for Reinforced Concrete Arch.—*Numerical Example.*

To illustrate further the analysis of the fixed arch by means of influence lines we will construct the influence diagrams for M_0, H_0, and V_0 (i.e., the supporting forces applied at the elastic center) for the arch shown in Fig. 168. The influence diagram for the moment at an intermediate point will also be drawn.

The method used varies slightly from that discussed in Art. 124.

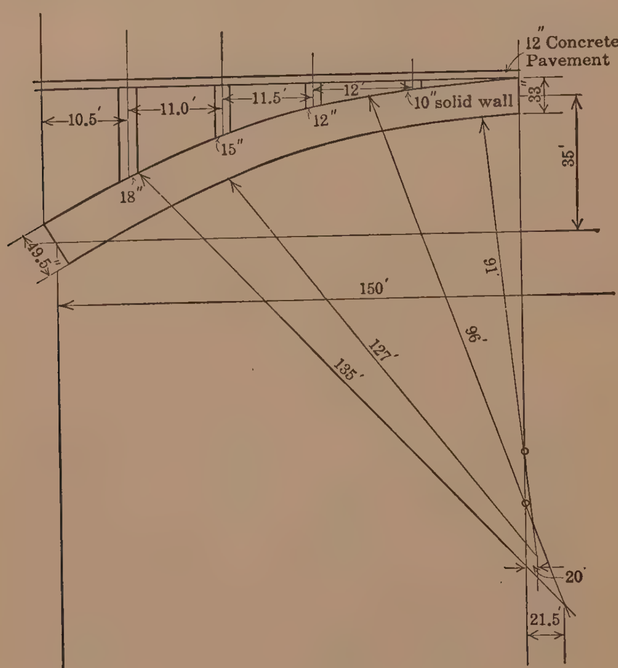


Fig. 168

The redundant forces are applied at the elastic center, but the arch is treated as a single cantilever (see Fig. 169*a*) instead of two symmetrical cantilevers.* The analysis to obtain H_0, M_0, and V_0 is shown in full detail in Table A and accompanying notes. The detail of dividing the arch ring to render $\frac{\Delta s}{I}$ constant, and the location of the elastic center, O, is omitted.

* See an article by C. S. Whitney, Engineering Record, Sept. 11, 1915. The method of evaluating the summations used in Table A was suggested by Mr. J. A. Wise.

Fig. 169 shows the influence diagram for H_0, V_0, and M_0, and also for M_5. The calculations for the latter are shown in the Table A.

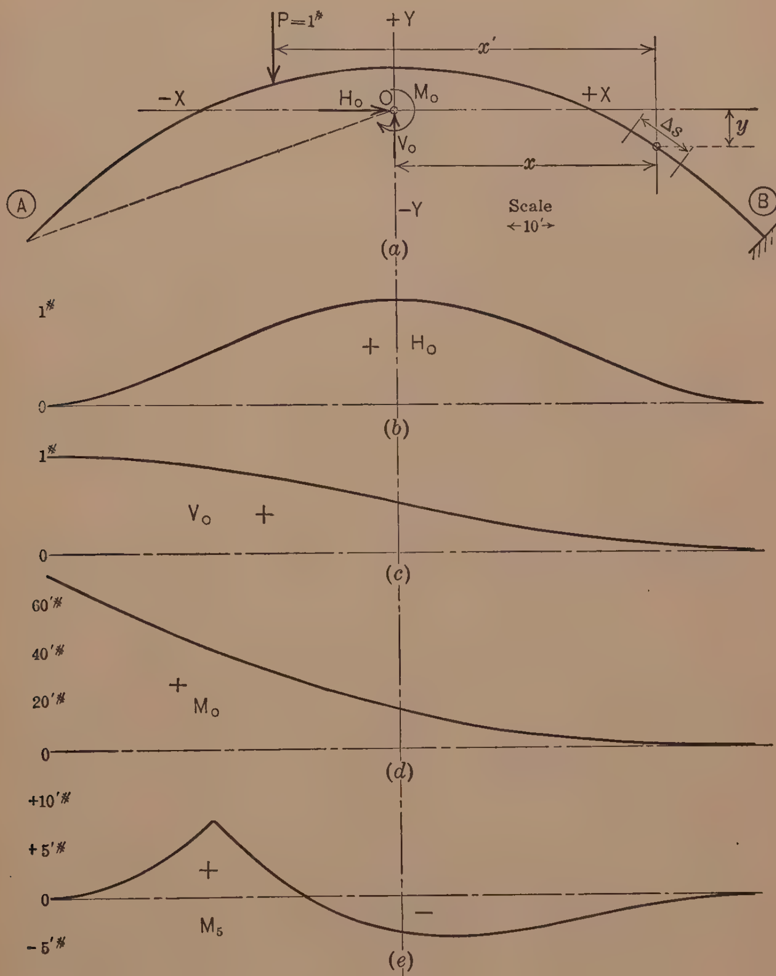


FIG. 169

TABLE A

Calculation of Influence Ordinates for V_0, H_0 and M_0

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
Pt.	x	Σx	Δx	$(\Sigma x)\cdot\Delta x$	$\Sigma xx' = \Sigma(\Sigma x)\cdot\Delta x$	x^2	$\frac{\Sigma x'x}{\Sigma x^2} = V_0$	y	Σy	$(\Sigma y)\cdot\Delta x$	$\Sigma yx' = \Sigma(\Sigma y)\Delta x$	y^2	$\frac{-\Sigma yx'}{\Sigma y^2} = H_0$	$\Sigma x'$	$\frac{\Sigma x'}{n} = M_0$	Pt.
S.L.R	75.0	....	....	....	....	....	....	....	....	....	....		....	....	....	S.L.R
1R	70.25	70.25	9.43	662.5	....	4935.0	....	−21.95	−21.95	−206.99	....	481.80	....	....	....	1R
2	60.82	131.07	8.82	1156.0	662.5	3699.0	.0191	−13.50	−35.45	−312.67	−206.99	182.25	0.1001	9.43	0.40	2
3	52.00	183.07	7.72	1413.3	1818.0	2704.0	.0525	−7.11	−42.56	−328.56	−519.66	50.55	0.2524	27.07	1.12	3
4	44.28	227.35	6.65	1511.9	3232.0	1961.0	.0934	−2.60	−45.16	−300.31	−848.22	6.76	0.4120	50.23	2.09	4
5	37.63	264.98	5.78	1531.6	4744.0	1416.0	.1370	0.65	−44.51	−257.27	−1148.53	.422	0.5580	76.83	3.20	5
6	31.85	296.83	5.36	1591.0	6275.0	1014.0	.1813	2.95	−41.56	−222.76	−1405.80	8.70	0.6829	105.73	4.41	6
7	26.49	323.32	5.04	1629.5	7866.0	702.0	.2272	4.60	−36.96	−186.28	−1628.56	21.16	0.7912	137.89	5.75	7
8	21.45	344.77	4.85	1672.1	9496.0	460.0	.2743	5.97	−30.99	−150.30	−1814.84	35.64	0.8816	173.17	7.22	8
9	16.60	361.37	4.78	1727.4	11168.0	276.0	.3226	6.94	−24.05	−114.96	−1965.14	48.16	0.9547	211.92	8.83	9
10	11.82	373.19	4.74	1768.9	12895.0	140.0	.3725	7.60	−16.45	−77.97	−2080.10	57.76	1.0105	254.99	10.57	10
11	7.08	380.27	4.72	1794.9	14664.0	50.1	.4236	8.05	−8.39	−39.60	−2158.07	64.80	1.0484	302.39	12.60	11
12R	2.36	382.63	4.72	1806.0	16459.0	5.8	.4757	8.38	0.0	0.0	−2197.67	70.22	1.0676	351.31	14.77	12R
12L	−2.36	380.27	4.72	1794.9	18265.0	5.8	.5277	8.38	8.39	39.60	−2197.67	70.22	1.0676	410.95	17.14	12L
11	....	373.19	4.74	1768.9	20060.0	50.1	.5795	....	16.45	77.97	....	....	....	472.45	19.68	11
10	....	361.37	4.78	1727.4	21829.0	140.0	.6306	....	24.05	114.96	....	....	....	538.65	22.45	10
9	....	344.77	4.85	1672.1	23557.0	276.0	.6805	....	30.99	150.30	....	....	....	610.15	25.42	9
8	....	323.32	5.04	1629.5	25229.0	460.0	.7288	....	36.96	186.28	....	....	....	687.75	28.67	8
7	....	296.83	5.36	1591.0	26858.0	702.0	.7758	....	41.56	222.96	....	....	....	768.45	32.25	7
6	....	264.98	5.78	1531.6	28449.0	1014.0	.8218	....	44.51	257.27	....	....	....	864.95	36.28	6
5	....	227.35	6.65	1511.9	29881.0	1416.0	.8661	....	45.16	300.31	....	....	....	973.45	40.80	5
4	....	183.07	7.72	1413.3	31493.0	1961.0	.9097	....	42.56	328.56	....	....	....	1106.45	46.30	4
3	....	131.07	8.82	1156.0	32906.0	2704.0	.9506	....	35.45	312.67	....	....	....	1278.45	53.15	3
2	....	70.25	9.43	662.5	34062.0	3699.0	.9840	....	21.95	206.99	....	....	0.1001	1473.95	61.20	2
1L	−70.25	0.0	....	....	34724.0	4935.0	1.0030*	−21.95	0.0	....	206.99	481.80	....	1680.95	70.10	1L
S.L.L	−75.0	....	....	....	....	....	....	−26.50	....	....	....	....	....	....	....	S.L.L

$\Sigma x^2 = 34620.0$ * Should equal unity. $\Sigma y^2 = 2058.44$

NOTES

(*a*) Δx as used in the table is for any section, r, $= x_r - x_{r+1}$. The remaining terms are fully defined in Fig. 169.

(*b*) With $\frac{\Delta s}{I}$ = constant, we shall have: (See Fig. 169) $V_0 = \frac{-\Sigma M'm}{\Sigma m^2} = \frac{\Sigma xx'}{\Sigma x^2}$; $H_0 = \frac{\Sigma M'm}{\Sigma m^2} = \frac{-\Sigma yx'}{\Sigma x^2}$; $M_0 = \frac{\Sigma M'm}{\Sigma m^2} = \frac{\Sigma x'}{n}$ (if n = number of divisions of arch-ring).

(*c*) In evaluating the above summations we note that for a load at pt. 2, taking $\Sigma xx'$ for illustration, $\Sigma_2 xx' = x_1(x_1 - x_2) = x_1\Delta_1 x$; for load at 3, $\Sigma xx' = x_1(\Delta_1 x + \Delta_2 x) + x_2\Delta_2 x = (x_1 + x_2)\Delta_2 x + x_1\Delta_1 x = \Sigma_2 xx' + (\Sigma_2 x)\cdot\Delta_2 x$. Thus in the table any value in column 6, say at pt. 4, is obtained by adding the values in columns 5 and 6 for pt. 3. The same general procedure is followed in evaluating $\Sigma yx'$. For $\Sigma x'$ we may write the general expression $\Sigma_{r+1}x' = \Sigma_r x' + r\Delta_r x$, i.e., for load at 3, $\Sigma_3 x' = \Sigma_2 x' + 2\Delta_2 x$, etc.

(*d*) It will be observed that for a unit load at 1_R, all summations vanish, while with the load at 1_L, all vanish except V_0, which becomes unity. That is to say the effective span is the distance between the centers of the extreme divisions of the arch-ring. This discrepancy is ordinarily of little importance.

(*e*) Noting that $M' = x'$ in all cases, we may write the equation for the moment at any point q as $M_q = M_0 - x'_q - H_0 y_q + V_0 x_q$. This equation is plotted in Fig. 169*e* for $q = 5$.

TABLE B

TABLE OF INFLUENCE ORDINATES FOR M_5

$$M_5 = M_0 - x'_5 - H_0y_5 + V_0x_5$$

Point	M_0	$-H_0y_5$	$+V_0(-x_5)$	$M' = -X'_5$	M_5
2R	.40	.065	.72	....	−0.39
3	1.12	.162	1.99	....	−1.03
4	2.09	.263	3.51	....	−1.68
5	3.20	.363	5.16	....	−2.32
6	4.41	.448	6.82	....	−2.86
7	5.75	.513	8.54	....	−3.30
8	7.22	.572	10.33	....	−3.68
9	8.83	.620	12.12	....	−3.91
10	10.57	.657	14.01	....	−4.10
11	12.60	.682	15.93	....	−4.01
12R	14.77	.693	17.86	....	−3.78
12L	17.14	.693	19.82	....	−3.37
11	19.68	.682	21.80	....	−2.80
10	22.45	.657	23.72	....	−1.93
9	25.42	.620	25.60	....	−0.80
8	28.67	.572	27.45	....	+0.65
7	32.25	.513	29.20	....	+2.54
6	36.28	.448	30.90	....	+4.87
5	40.80	.363	32.60	....	+7.84
4	46.30	.263	34.20	6.65	+5.19
3	53.15	.162	35.75	14.37	+2.87
2	61.20	.065	37.05	23.19	+0.89
1L	70.25	0	37.63	32.62	0

Values for 1_R are negligibly small.

CHAPTER VII

SECONDARY STRESSES

132. General Discussion.—The assumptions underlying the ordinary computation of stresses in trusses * are that the members are connected at the joints by frictionless pins placed exactly in the gravity axis of each member, and that the applied loads are concentrated at the joints. Under such conditions, the only stress developed in any structural member would be direct axial tension or compression. It is rare that these conditions are fully realized for every member of a truss, or even closely approximated, but it is true that on the average, the axial stresses calculated on this basis are much the largest and most important stresses that a member of a truss has to resist, and they are almost universally designated as the *primary* stresses.

Other stresses do occur—quite commonly in some members. We may mention the bending stresses due to weight in a horizontal member, and similar stresses due to applied loads between joints; bending due to portal action in the end posts of a bridge and that due to floor beam deflection in the intermediate posts; the bending due to pin eccentricity in any member, and the bending in the plane of the truss induced at the joints when a truss is built with riveted instead of pin connections.

Stresses due to any of the preceding causes may, under special conditions, become very large and of the same relative importance as the primary stresses. Generally, they are, as stated, much smaller and less important, and the whole group is included under the term "secondary" stresses, when used in the broadest sense. However, the last named division of the group, i.e., the bending stresses in the plane of the truss due to rigid joint connections, stands in a class quite by itself, due to the wide occurrence of such stresses, the difficulty of their calculation, and their importance in the design of many types of long-span bridges. It has therefore become more or less common practice to apply the term "secondary stress" in the narrower sense exclusively to this type. It will be so used in this chapter except as otherwise noted. Stresses of

* It is understood that the term "truss" as here used is not meant to include any framework where rigidity of some or all joints is a necessary condition for structural stability. Such structures are technically classed as "frames."

this type will occur in every truss where some or all the joints are riveted, or where the assumed "frictionless" pin is not actually frictionless. Speaking broadly, we may say that secondary stresses are of small importance in light trusses, even when fully riveted, but that in heavy riveted trusses, they frequently attain an order of magnitude somewhat

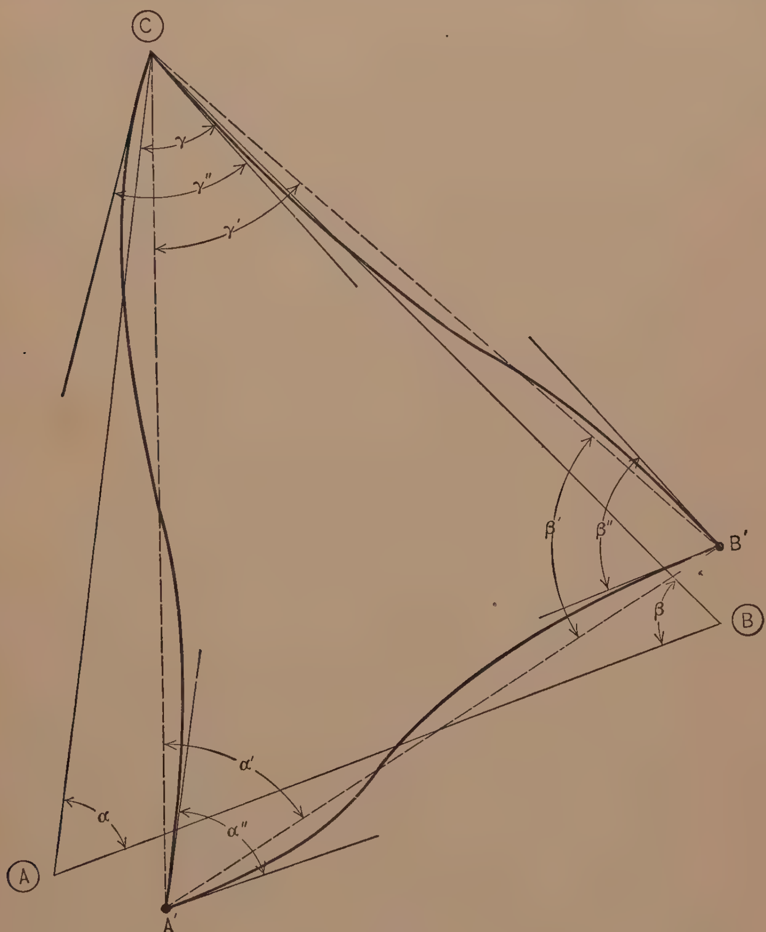


Fig. 170

comparable to the primary stresses (30 per cent to 60 per cent), in occasional rare cases becoming of equal importance.

133. Nature of Problem.—The physical phenomena giving rise to secondary stresses may be visualized qualitatively by means of a greatly exaggerated sketch such as Fig. 170. *ABC* is a triangular frame—either

independent or an element of a truss. Through distortion of the members, the frame takes the form $A'B'C$, the angles α, β, γ becoming α', β', γ', on the assumption of smooth pins at each joint. If, instead, the joints are rigid, then the end tangents must maintain a constant angle between them so that $\alpha = \alpha''$; $\beta = \beta''$; $\gamma = \gamma''$. Then the distortion cannot take place, in general, except by bending the members as shown.

If we attack the secondary stress problem by means of the general theory of indeterminate stresses, we observe that a truss with rigid joints is statically equivalent to the same truss with pin joints and moments applied to the ends of each member sufficient in any case to maintain a constant angle between the various members at any joint. Regarding the pin truss as the statically undetermined base system, we shall have $2m$ applied end moments, if m equal the number of members in the truss. To determine these moments, we have at every joint one statical equation, in virtue of the fact that $\Sigma M = 0$, and $k - 1$ elastic equations, if $k =$ the number of members entering any given joint. These latter arise from the fact that when the truss deflects under load, each joint must rotate *as a whole* (if at all), in view of the assumption of rigid joint connections. Calling this rotation ϕ and the rotation of the end tangents of the individual members entering the joint $\phi_1, \phi_2 \ldots \phi_k$, we must have

$$\phi_1 = \phi_2 = \ldots \phi_k = \phi.$$

Thus, if we have two members at a joint, we have one elastic (deflection) equation, viz., $\phi_1 = \phi_2$; if we have three members, we have two such equations—$\phi_1 = \phi_2 = \phi_3$, and for k members, we obviously have $k - 1$ equations. Now, it is evident that Σk for all joints $= 2m$, and therefore $\Sigma(k - 1) = 2m - n$. Thus, since we have $2m$ unknown moments, and n statical relations and $2m - n$ elastic relations, we may always solve the problem. Assuming the truss statically determinate and stable, the necessary relation between m and n is $m = 2n - 3$, whence $2m - n = 3(n - 2)$.

Recalling that in a pin-connected truss, the ϕ's for the individual member-ends will in general be different, the process of solution is clear: We determine by well-known methods these various angle changes (due to given loads) in the pin truss, and by means of the $2m - n$ relations, we determine the necessary moments to apply to make these values equal at each joint. With these values determined, the correct final primary stresses can be computed.

Such a process, though simple enough as to theory, would be appallingly laborious to carry through. An elementary structure such as

Fig. 171 is 9-fold indeterminate; a simple 6-panel Pratt truss is 30-fold indeterminate.

A further complication in effecting an exact solution should be noted. It is evident from Fig. 172 that bending of the member mn will be a function of the direct stress S as well as of the applied end moments, M_m and M_n, since due to its own deflection, there will exist at any section of the member a moment $S \times y$ in addition to the ordinary beam bending moment. This fact does not add to the degree of statical indetermination, but it vastly complicates the elastic equations. To illustrate, in the process of determining the final rotation of the end tangent

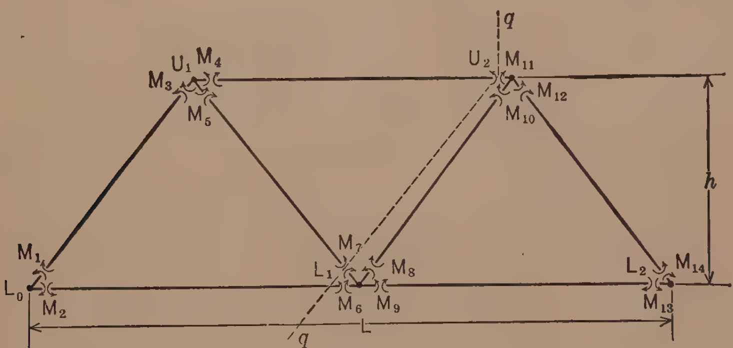


FIG. 171

of a given member, we need the angle τ of Fig. 172 resulting from the applied end moments. We may obtain this as

$$\tau_m = \int_n^m \frac{M m dx}{EI}, \quad \ldots \ldots \quad (91)$$

where "m" is the moment at any section of the member due to a dummy unit couple at m, and M the actual bending moment at any section. Neglecting the effect of direct stress, we may write

$$M = M_m - V_m x = M_m - \frac{M_m + M_n}{L} \cdot x. \quad \ldots \quad (92)$$

If, however, we consider the effect of the axial stress, this becomes

$$M = M_m - \frac{M_m + M_n}{L} \cdot x + Sy. \quad \ldots \ldots \quad (93)$$

Suppose we consider the member U_1U_2 of Fig. 171. The correct stress S will no longer be obtained by a simple computation of moments of loads and reactions about L_2; it will involve the effect of the moments

M_6, M_7, M_{11} (if we take a section $q \ldots q$ as shown) and the corresponding shears (not shown in figure) at the right ends of the members cut. These shears in turn will be expressed in terms of the two end moments; —thus for U_1U_2

$$V_{11} = \frac{M_4 + M_{11}}{L}, \text{ if } L = \text{length of } U_1U_2.$$

It will be seen then that the algebraic expression for S will involve the ordinary effect of the loads, which we may write as ΣPC, when C (for each load) is a proper constant, and also the *six* bending moments.

The exact expression for y for a member subjected to direct stress as well as bending will be obtained from an integration of the differential equation,

$$EI\frac{d^2y}{dx^2} = -M = -M_m + \frac{M_m + M_n}{L}x - Sy. \quad \ldots \ldots \quad (94)$$

This integrates into

$$y = C_1 \cos\frac{x}{K} + C_2 \sin\frac{x}{K} - \frac{1}{S}\left(M_m + \frac{M_m + M_n}{L}x\right), \quad . \quad (95)$$

where $K = \frac{EI}{L}$, and C, and C_2 are constants.

In order, then, to obtain a precisely correct value for the rotation of the end tangent, it is necessary to obtain a general expression for S as indicated, and substitute it in equation (95) to obtain an explicit expression for y; from this M for any point of the beam may be expressed by means of equation (93) and the angle change finally expressed from equation (91). It will be observed that the "law of proportionality" does not apply to moments (and resulting stresses) expressed by such equations as (93). The term y is a function of S and the M's, therefore the equation is no longer linear in terms of the loading, and a complete independent calculation is required for every different load condition.

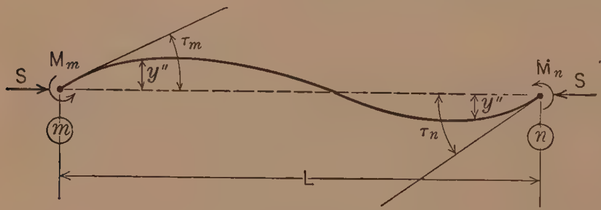


FIG. 172

This discussion is intended to make clear the point that secondary stresses may be analyzed by the same general method of pro-

cedure applicable to other indeterminate stress problems, but that such an analysis is far too complex and unwieldy to be useful as a working method. Since the methods commonly in use for computing secondary stresses may seem to the student radically different from the fundamental methods used as a basis for solution of other indeterminate problems, it seems worth while to note carefully the simplifying assumptions usually made, their justification, and their effect in modifying the procedure.

134. Simplifying Assumptions.—(*a*) It is clear from Fig. 171, that, taking moments about L_2, the moment of the external forces is resisted by the direct stress in U_1U_2 and by the moments and shears at the ends of the three members cut by section $q - q$. Omitting the shears from consideration (since their effect is usually negligibly small compared to that of the other quantities) the question arises, what proportion of the external moment is resisted by the axial stress in such a member as U_1U_2 and what by the end moments? A general explicit answer to the question is impossible, but some idea of the upper limits may be obtained from the following reasoning:

If h = height of truss (or moment arm of given chord member);
d = depth of chord member considered;
f_P = primary unit stress;
f_S = secondary unit stress,

and I, A and c have their usual meaning, then we may write

$$\begin{aligned} M_P &= \text{resisting moment due to primary stress} \\ &= Sh = f_PAh, \text{ and} \\ M_S &= \text{resisting moment due to secondary bending} \\ &= f_S\frac{I}{c} = \frac{f_SAr^2}{\frac{d}{2}}, \end{aligned}$$

$$\frac{M_S}{M_P} = \frac{\frac{f_SAr^2}{\frac{d}{2}}}{f_PAh} = \frac{f_S}{f_P}\cdot\frac{2r^2}{hd} = \frac{f_S}{f_P}\times .32\frac{d}{h},$$

if we assume r = approximately .4 d.

If now we assume as an extreme limit that $d = \frac{h}{10}$ and $f_S = f_P$, we find that the secondary bending in the given chord will furnish about 3.2 per cent as much resistance to external moment as the primary

stress. The lower chord and the diagonal as commonly built will have a much smaller relative effect, so that even under the extreme conditions considered, it is doubtful if the secondary moments can furnish as much as 5 per cent of the resistance—that is to say that the exact primary stress is 95 per cent or more of the primary stress computed in the ordinary manner.

Of course, the assumptions just made are very extreme. The ordinary ratio of depth of member to depth of truss is seldom more than $\frac{1}{15}$ even for heavy riveted trusses, while no well-designed bridge is likely to exhibit secondary stresses equal to 100 per cent * of the primary. The ordinary provision made in office designing is for a secondary of 15 per cent to 25 per cent of the primary; a secondary stress amounting from 50 per cent to 60 per cent is to be regarded as *very* high. A comparison made by Professor Turneaure † for a 6-panel Pratt truss showed a reduction of the primary stress from secondary bending ranging from $\frac{1}{3}$ to $\frac{1}{2}$ of 1 per cent for typical chord and web members. A similar comparison made for the very large Kenova truss (see Fig. 176), indicates that for simultaneous secondary stresses in all contributing members of 60 per cent of primary, the reduction of primary stress in the top chord members would be but slightly over 1 per cent.

The significance of this fact from the viewpoint of analysis cannot well be overestimated, for it means that the primary stresses are, for ordinary types of bridge trusses, small and large, practically unaffected by the secondary stresses, and *may be computed independently and in advance of the latter.*

The foregoing brief discussion of the *general* method of attack on the secondary stress problem is sufficient to indicate to what a vast simplification the assumption of the independence of the primary stresses leads. This assumption is always made ‡ in developing the theory of

* It is hardly necessary to point out that when we speak of a secondary stress equal to a certain per cent of the primary, we are comparing the unit stress on the extreme fiber due to secondary bending with the axial unit stress uniformly distributed over the section.

† Johnson, Bryan and Turneaure, "Modern Framed Structures," Part II, page 500–501.

‡ Statements are sometimes made that riveted trusses are stiffer than pin trusses and that the assistance rendered by rigid joints in carrying the truss loads tends automatically to reduce the secondary stresses as computed. In so far as such statements mean that secondary stresses appreciably relieve the primary stresses, the foregoing arguments indicate that they are without foundation in fact. There are, however, important individual exceptions. For a fuller discussion, see Maney and Parcel, University of Minnesota, Studies in Engineering No. 4, "An investigation of Secondary Stresses in the Kenova Bridge."

secondary stresses; so far as is known no attempt has been made to analyze such stresses by the general method outlined in Art. 133, even as a matter of purely theoretical interest.

(*b*) The effect of the primary stress on the bending of a member may become very considerable for long slender members, but generally speaking, it is for just such members that secondary stresses are the least serious. For the stocky type of compression member ordinarily met with in heavy riveted trusses (and it is for this type chiefly that secondary stresses are important), the effect of direct stress in modifying the secondary bending is seldom more than 6 per cent, and usually much less. If we make the simplifying assumption (*a*) it is quite possible to develop an otherwise exact solution, taking account of the effect of the primary stress on the bending, in a manageable form. As a matter of fact, the original solution * of the secondary stress problem was of this type. It is more involved in theory and laborious in application than the approximate method (to be later described) now in general use, and authorities are generally agreed that the gain in accuracy is not worth the effort expended.† In the case of secondary stresses, as with many other statically indeterminate problems, while we may feel that the theory is dependable and correct to a reasonable degree of approximation, we must recognize that any refinement of calculation is futile.

It is the universal rule of engineering practice in America and in Europe to make both the simplifying assumptions just discussed, i.e., that (*a*) the primary stresses are independent of the secondary stresses and (*b*) that the effect of the direct stress in modifying the bending is negligibly small. With these assumptions, the solution is easily effected by a direct application of the slope-deflection method as indicated in the following article.

135. Application of Slope-deflection Equations.—We recall from Chapter III that for any bar subjected to bending, whether an independent beam or a member of a framework, the correct end moment for any condition of support may be expressed as

$$M_{mn} = M_{Fmn} - 2\frac{EI}{L}\left(2\phi_m + \phi_n - \frac{3D}{L}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

* Due to H. Manderla, 1879. See Allgemeine Bauzeitung, 1880—"Die Berechnung der Sekundärspannungen in einfachen Fachwerk infolge starrer Knotverbindungen."

† For a very thorough discussion of the "exact" solution (as Manderla's solution is usually called), see Johnson, Bryan and Turneaure, "Modern Framed Structures," Part II, pages 507–521.

where

$M_{F_{mn}}$ = the end moment at m in the beam mn if the ends are fully fixed;

$\phi_m, \ldots \phi_n$* = the rotation of the end tangent from the original position at m and n respectively;

D = the relative linear deflection of one end of the beam with respect to the other, measured normal to the axis of the beam, and E, I and L have their usual significance.

In all ordinary bridge trusses, the loads are applied at the joints only, hence $M_{F_m} = 0$. Also, we may take E as sensibly constant, and it will then be much more convenient to use $E\phi$ and ED than the true ϕ and D. Unless specifically noted otherwise, we shall in the remainder of this chapter so use these symbols. The equation then becomes

$$M_{mn} = \frac{2I}{L}\left(2\phi_m + \phi_n - \frac{3D}{L}\right). \quad . \quad . \quad . \quad . \quad . \quad (97)$$

Assuming rigid joints, we must have the same rotation of the end tangent for all members entering a joint; we therefore have n and only n independent values of ϕ, if n is the number of joints. In virtue of the simplifying assumption (a), the values of D may be computed prior to the secondary stress solution. For it is a fundamental principle of structural mechanics that within the distortion limits comprehended by the elastic theory, the bending of a member does not change its axial length. (The student will recall that in developing the equation of the elastic curve in Strength of Materials, it is assumed that $ds = dx$.) It is further an elementary principle of geometry that the distortion of a jointed framework composed of triangular elements is a function of the change of lengths of these members and nothing else. Since the length is not affected by flexure, we must conclude that axial distortion due to the primary stresses is the sole agency in producing deflection. We are assuming that these stresses may be computed in the usual way, quite independent of the secondaries, hence the values of D are predetermined. They are most simply obtained from a Williot diagram. For any given loading, the values of D may all be scaled from a single diagram. Since D thus becomes a known quantity, we have a solution completely made out for all the unknown bending moments by means of equation (97), so soon as the values of ϕ are known. To determine these n values of ϕ, we have n statical equations in virtue of the fact that

* The symbol Θ was used for this quantity in Chapter III.

$\Sigma M = 0$ for every joint. Considering a joint with four members (see Fig. 173) we have

$$M_{mn} = -\frac{2I_{mn}}{L_{mn}}\left[2\phi_m + \phi_n - \frac{3D_{mn}}{L_{mn}}\right],$$

$$M_{mo} = -\frac{2I_{mo}}{L_{mo}}\left[2\phi_m + \phi_o - \frac{3D_{mo}}{L_{mo}}\right],$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$M_{mq} = -\frac{2I_{nq}}{L_{nq}}\left[2\phi_m + \phi_q - \frac{3D_{mq}}{L_{mq}}\right].$$

Calling $\frac{D}{L} = R$ and $\frac{I}{L} = K$,

$$\sum_n^q M = 0 = 2\sum_n^q K\phi_m + K_{mn}\phi_n + K_{mo}\phi_o + K_{mp}\phi_p$$
$$+ K_{mq}\phi_q - 3\sum_n^q KR = 0. \quad . \quad . \quad . \quad . \quad (98)$$

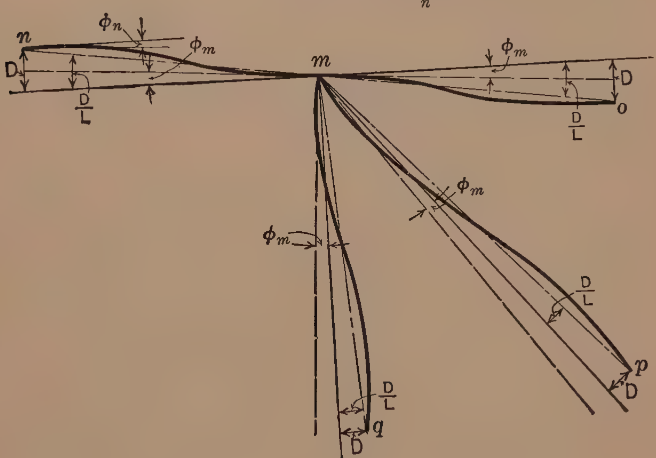


FIG. 173

The solution of any problem in secondary stresses by slope-deflections consists in the setting up of n such equations as (98) and their simultaneous solution.*

136. *Example.*—We shall first consider a very simple problem—the six-panel Pratt truss † of Fig. 174 with a single load at the center.

* This solution of the secondary stress problem is originally due to O. Mohr—"Die Berechnung der Fachwerke mit starrer Knotverbindungen."—Der Ziviling, 1892, page 577; 1893, page 67.

† This is fully analyzed by a different method in Johnson, Bryan and Turneaure, "Modern Framed Structure," Part II, pages 441 et seq.

The stresses and total deformations are shown in the figure, and the lengths and section constants in table A. Fig. 175 shows the Williot diagram, drawn assuming 6–7 to stand fast. The values of D (actually $E \times$ deflection) are shown on this diagram. Table A contains all the data necessary to set up equations similar to (98) for each joint, and these are shown at the head of table B. For joint 1 (L_0) for example, equation (98) becomes

$$2\Sigma K \cdot \phi_1 + K_{1-2}\phi_2 + K_{1-3}\phi_3 = K_{1-2} \cdot 3\frac{D_{1-2}}{L_{1-2}} + K_{1-3} \cdot 3\frac{D_{1-3}}{L_{1-3}}.$$

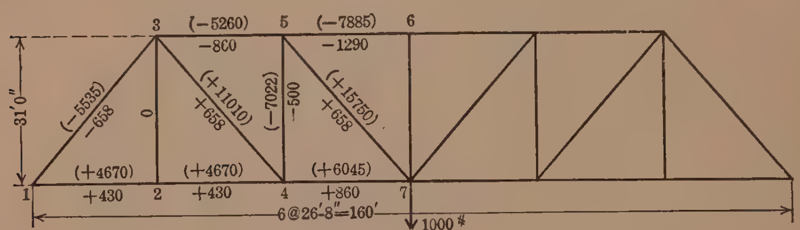
Substituting numerical values from table A, we get

$$2(3.80 + 9.15)\phi_1 + 3.80\phi_2 + 9.15\phi_3 = 1145 + 2384,$$

or

$$25.90\phi_1 + 3.80\phi_2 + 9.15\phi_3 = 3529,$$

which is the first equation in table B. The remainder of table B show



NOTE:—Quantities in parenthesis = total deformation $\times E$.
Quantities written below members = total stress.

FIG. 174

the detail of the simultaneous solution and requires no special comment. Since the secondary bending moment is

$$M_{mn} = -\frac{2I_{mn}}{L_{mn}}\left(2\phi_m + \phi_n - 3\frac{D_{mn}}{L_{mn}}\right),$$

the secondary stress must be

$$f_s = M_{mn} \cdot \left(\frac{C}{I}\right)_{mn} = -2\left(\frac{C}{L}\right)_{mn}\left(2\phi_m + \phi_n - 3\frac{D_{mn}}{L_{mn}}\right).$$

The third column of table C shows the stresses so computed. The compression chords, having different values of c top and bottom, have two values of the secondary stress.

A special property of the secondary stress equations will be observed in table B. The coefficient along the diagonal line—i.e., the coefficients of ϕ at the joint for which the equation is written—is very much larger than the other coefficients. This at once suggests an approximate method of solution. It is evident if we have a very crudely approximate set of values for ϕ, estimated or obtained in any other manner, and if these values are successively substituted in each equation *in the terms with small coefficients*, and the equations solved for the ϕ with the large coefficient, we shall by this process obtain a much closer approximation to the true values. We shall try a very rough assumption—that all values of ϕ are equal. Then, for ϕ_5 for example, if we assume ϕ_3 and ϕ_4 to have the *same* value, joint equation 5 may be written,

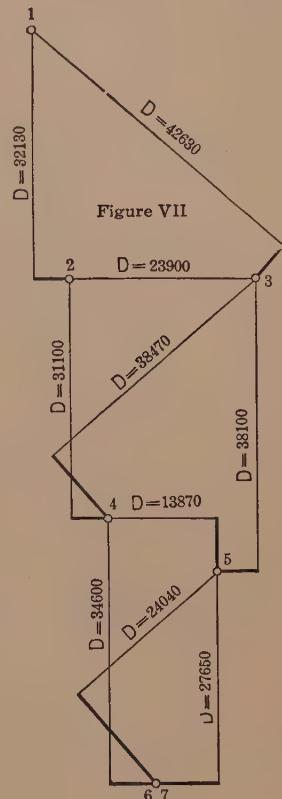


FIG. 175

$$\phi_5(55.2 + 2.02 + 12.4) = 7993,$$

and the first approximation for ϕ_5 is

$$\phi_5 = \frac{7993}{69.67} = 114.6,$$

TABLE A

Member	Section Area Sq. Inches	Length Inches	I	c Inches	$\frac{I}{L}$	$\frac{3D}{L}$	$\frac{3ID}{L^2}$
1–2	29.44	320	1218	9.12	3.80	301.2	1145
1–3	58.49	490.7	4490	9.54 14.08	9.15	260.5	2384
2–3	16.00	372	94.8	5.4	.255	192.8	49
2–4	29.44	320	1218	9.12	3.80	291.6	1107
3–4	29.42	490.7	805	7.5	1.60	213.3	350
3–5	52.35	320	3978	9.19 14.43	12.43	357.3	4440
4–5	26.48	372	750	7.5	2.02	111.8	226
4–7	45.48	320	1907	9.12	5.96	324.3	1932
5–6	52.35	320	3978	9.19 14.43	12.43	259.3	3220
5–7	20.58	490.7	358	6.0	.731	147.1	107
6–7	14.70	372	288	6.0	.774	0	0

TABLE B

SOLUTION OF SIMULTANEOUS EQUATIONS FOR SLOPE VALUES AT JOINTS OF TRUSS

Equation For	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	Constant Term
Joint 1	25.90	3.80	9.15	....	....	3529
Joint 2	3.80	15.71	.255	3.80	....	2301
Joint 3	9.15	.255	46.95	1.64	12.43	7223
Joint 4	....	3.80	1.64	26.84	2.02	3615
Joint 5	....	....	12.43	2.02	55.22	7993
A	1	.1468	.3533	....	....	136.2
B	1	4.132	.0671	1	....	605.0
C	1	.0279	5.130	.1793	1.359	789.0
A-B	....	−3.9852	.2862	−1	....	−468.8
B-C	....	4.1041	−5.0629	.8207	−1.359	−184.0
6	....	1	− .0718	.251	....	117.7
7	....	1	−1.2345	.200	−33.12	− 44.8
D	....	1	.4316	7.06	.5315	951.0
6-7	....	....	1.1627	.051	.3312	162.5
7-D	....	....	−1.6661	−6.86	− .8627	−995.8
8	....	....	1	.0438	.2847	139.8
9	....	....	1	4.118	.5180	597.8
E	....	....	1	.1625	4.440	643.0
8-9	....	....	....	−4.074	− .2333	−458.0
9-E	....	....	....	3.9555	−3.922	− 45.2
10	....	....	....	1	.05725	112.4
11	....	....	....	1	− .9915	− 11.4
10-11	....	....	....	....	1.0487	123.8
ϕ_5	....	....	....	....	....	118.2
	....	....	....	....	....	112.4
		−	.05725 (118.2)			− 6.8
ϕ_4	....	....	....	....	....	105.6
	....	....	....	....	....	139.8
		−	.0438 (105.6)			− 4.62
		−	.2847 (118.2)			− 33.67
ϕ_3	....	....	....	....	....	101.5
	....	....	....	....	....	117.7
		+	.0718 (101.5)			7.3
	....	....	....	....	....	125.0
		−	.251 (105.6)			− 26.5
ϕ_2	....	....	....	....	....	98.5
	....	....	....	....	....	136.2
		−	.1468 (98.5)			− 14.46
		−	.3533 (101.5)			35.84
	....	....	....	....	....	50.3
ϕ_1	....	....	....	....	....	85.9

which is practically in agreement with the true value, 118.2. The tabulation below shows the very striking agreement throughout.

ϕ	Exact	1st Approximation
ϕ_1	86	91
ϕ_2	98.5	98.5
ϕ_3	101.5	103.0
ϕ_4	105.6	108.8
ϕ_5	108.2	114.6

TABLE C

Location	$\frac{2c}{L}$	$\frac{3D}{L}-2\phi_m-\phi_n$	Fiber Stress	$\frac{3D}{L}-2\phi_m-\phi_n$ (Approx.)	Fiber Stress by Approximate Method
1–2	.057	+ 30.9	+ 1.76	+ 21.8	1.24
2–1	.057	+ 18.3	+ 1.04	+ 15.1	.86
1–3	.0388	− 12.9	− .50	− 24.0	− .93
	−.0574	− 12.9	+ .74	− 24.0	1.38
3–1	.0388	− 28.4	− 1.63	− 35.8	− 2.06
	−.0574	....	+ 1.103	....	1.39
2–3	.029	−105.7	− 3.06	−105.1	− 3.05
3–2	.029	−108.7	− 3.15	−110.2	− 3.19
2–4	.057	− 11.0	− .62	− 8.9	− .51
4–2	.057	− 18.1	− 1.03	− 16.6	− .95
3–4	.0306	− 95.3	− 2.91	− 96.4	− 2.95
4–3	.0306	− 99.4	− 3.04	−100.0	− 3.06
3–5	.0574	+ 36.1	+ 2.07	+ 37.1	+ 2.13
	−.0902	....	− 3.26	....	− 3.35
5–3	.0574	+ 19.4	+ 1.75	+ 25.0	+ 2.25
	−.0902	....	− 1.136	....	− 1.43
4–5	.0404	−217.6	− 8.79	−213.6	− 8.63
5–4	.0404	−230.2	− 9.30	−223.1	− 9.0
4–7	.057	+113.1	+ 6.45	+113.7	+ 6.48
7–4	.057	+218.7	+12.46	+219.0	+12.50
5–7	.0244	− 89.3	− 2.18	− 82.5	− 2.02
7–5	.0244	+ 28.9	+ .705	32.3	.79
5–6	.0574	+ 22.9	+1.315	29.7	1.72
	−.0902	....	− 2.07	....	− 2.68
6–5	.0574	+141.1	+12.75	144.5	+13.00
	−.0902	....	− 8.10	....	− 8.30

The final stresses from these approximate values are shown in the last column of table C. For all the larger values of the stresses, the agreement is practically exact.

The stress equation is:

$$f = \text{constant} \times \left(\frac{3D}{L} - 2\phi_m - \phi_n\right), \quad . \quad . \quad . \quad . \quad (99)$$

when $\frac{D}{L}$ is correctly predetermined. It will be obvious that in all cases where $\frac{D}{L}$ and ϕ are of opposite sign and hence all the terms become additive, or wnen as here, the terms are of the same sign and $\frac{D}{L}$ is much larger than either of the values of ϕ, the final stresses will converge more rapidly by approximation than the values of ϕ themselves—otherwise the reverse will be true.

This must obviously be true since the value of the stress depends upon $\phi - R$ rather than ϕ alone. As noted, for the larger secondaries table C shows that the convergence of the actual stresses in the process of successive substitution is extremely rapid.

It cannot be assumed by any means that such rapid convergence toward the true values is obtainable by this method in every problem in secondary stresses. As a matter of fact, it is only by carrying through a second approximation that any adequate idea can be gained of how closely correct the results are. Thus, in the above example, if the solution had been carried out solely by the method of "successive approximations" or "successive substitutions," the procedure would involve a substitution of the set of values obtained by the first approximation just explained in each equation, thus determining a second set of approximate values. In this second set of substitutions, the first set of approximate values are substituted in the terms with small coefficients and the value of the ϕ with the large coefficient is obtained. Thus to find ϕ_1, we use the first equation, substituting the approximate values for ϕ_2 and ϕ_3,

$$25.9\phi_1 + 3.8 \times 98.5 + 9.15 \times 103 = 0,$$

whence

$$\phi_1 = \frac{3529 - 1316}{25.9} = 85.8.$$

Likewise to obtain, say, ϕ_4, we substitute the approximate values of ϕ_2, ϕ_3 and ϕ_5 in the fourth equation and solve for ϕ_4

$$26.8\phi_4 + 3.8 \times 98.5 + 1.64 \times 103 + 2.02 \times 114.6 = 3615,$$

and

$$\phi_4 = \frac{2840.6}{26.8} = 105.8.$$

Precisely the same process would be followed in case a third substitution were necessary. The smallness of the difference between successive sets of values for ϕ determines how close the approximation is to the true values.

The advantages of this method of solution of simultaneous equations will be further discussed in a later article. We may observe here that it is not applicable to *any* set of simultaneous linear equations; its successful application requires that one of the unknowns shall have a relatively large coefficient (and preferably that the unknowns shall at least be somewhere near the same order of magnitude). If we write the fourth joint equation divided by the coefficient of ϕ_4 we have

$$\phi_4 + .142\phi_2 + .061\phi_3 + .075\phi_5 = 135.$$

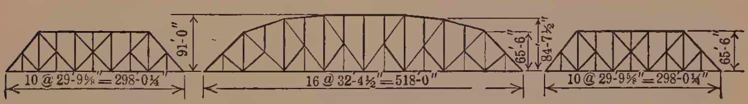


FIG. 176

It is clear that any errors in ϕ_2, ϕ_3 and ϕ_5 are greatly minimized by being multiplied by coefficients that are such small fractions. Unless these values are much larger than ϕ_4, it is clear that they may be in error by a considerable percentage without any great effect on the value of ϕ_4.

We must further note that the method is not available unless a crude first approximation for the unknowns is readily obtainable. This will generally be true in the analysis of secondary stresses. In the nature of the case there will be a degree of uniformity in the joint twists ϕ in a given region for ordinary loadings. For special cases of loading, special artifices must be resorted to, but these will usually suggest themselves to the computor.

137. *Example* 2.—We shall take for a second example a very different type of structure—a long-span Petit Truss. The structure used for illustration is the 518 ft. span of the Norfolk and Western Ry. bridge over the Ohio river at Kenova, W. Va. (see Fig. 176). It is selected because it is the prevailing type of bridge for simply supported spans exceeding 250–300 ft. in length and because it is a type in which the secondary stresses are likely to take high values and play a rather important

role.* Incidentally, it may be noted that the Kenova truss is one of the heaviest simple-span trusses with fully riveted joints that has ever been built.

The solution of the simultaneous equations will be carried through by a method of successive approximations somewhat different from that of the previous article. The entire solution, including all detail except slide-rule calculations, is contained on the single stress sheet shown in Figs. 177*a*, 177*b* and Table A. The method of approximation used and the arrangement of the calculations give to this solution an especial compactness and elegance.† It is believed that the stress sheet of Fig. 177*a* is largely self-explanatory, but the following additional notes may aid the student in following the calculations.

(1) Fig. 178*a* shows the relations between ϕ, R, δ and δ' if m is the joint of reference.

(2) The second equation in the list of formulae, viz.,

$$M = K(\delta + \tfrac{1}{2}\delta') = \frac{4EI}{L}(\delta + \tfrac{1}{2}\delta'),$$

may be readily reduced to the standard slope-deflection equation with which the student is familiar. Thus we have, substituting the values of δ and δ' in terms of ϕ, ϕ' and R,

$$M = \frac{4EI}{L}(\delta + \tfrac{1}{2}\delta') = \frac{4EI}{L}\left(\phi - R + \frac{\phi' - R}{2}\right) = \frac{2EI}{L}(2\phi + \phi' - 3R).$$

(3) The assumption made for a first approximation is that $\phi' = R$ (Fig. 178*b*). The student should understand clearly that this is a pure assumption, made to simplify the calculation so as to obtain a rough value of ϕ, just as in example 1 it was assumed for a first approximation that all values of ϕ were equal. *Any such assumption is to be justified by whether or not the value of ϕ so calculated is near enough the true value to be a useful approximation.* An examination of the stress sheet of Fig. 177*a* shows that not only is this much true for the problem under consideration, but that in many cases the first approximation is a satisfactory *final* value.

To assume $\phi' = R$, or $\delta' = 0$, is to assume that the moment at a joint depends entirely on the relative defections of the ends of the member

* 1921 A.R.E.A. Specifications recommend that the secondary stresses be actually calculated in this type of truss.

† For the method and detail of the solution, the authors are indebted to Mr. Allston Dana, C. E.· Except for some unimportant modifications in notation, the stress sheet of Fig. 177 and accompanying chart is entirely Mr. Dana's work.

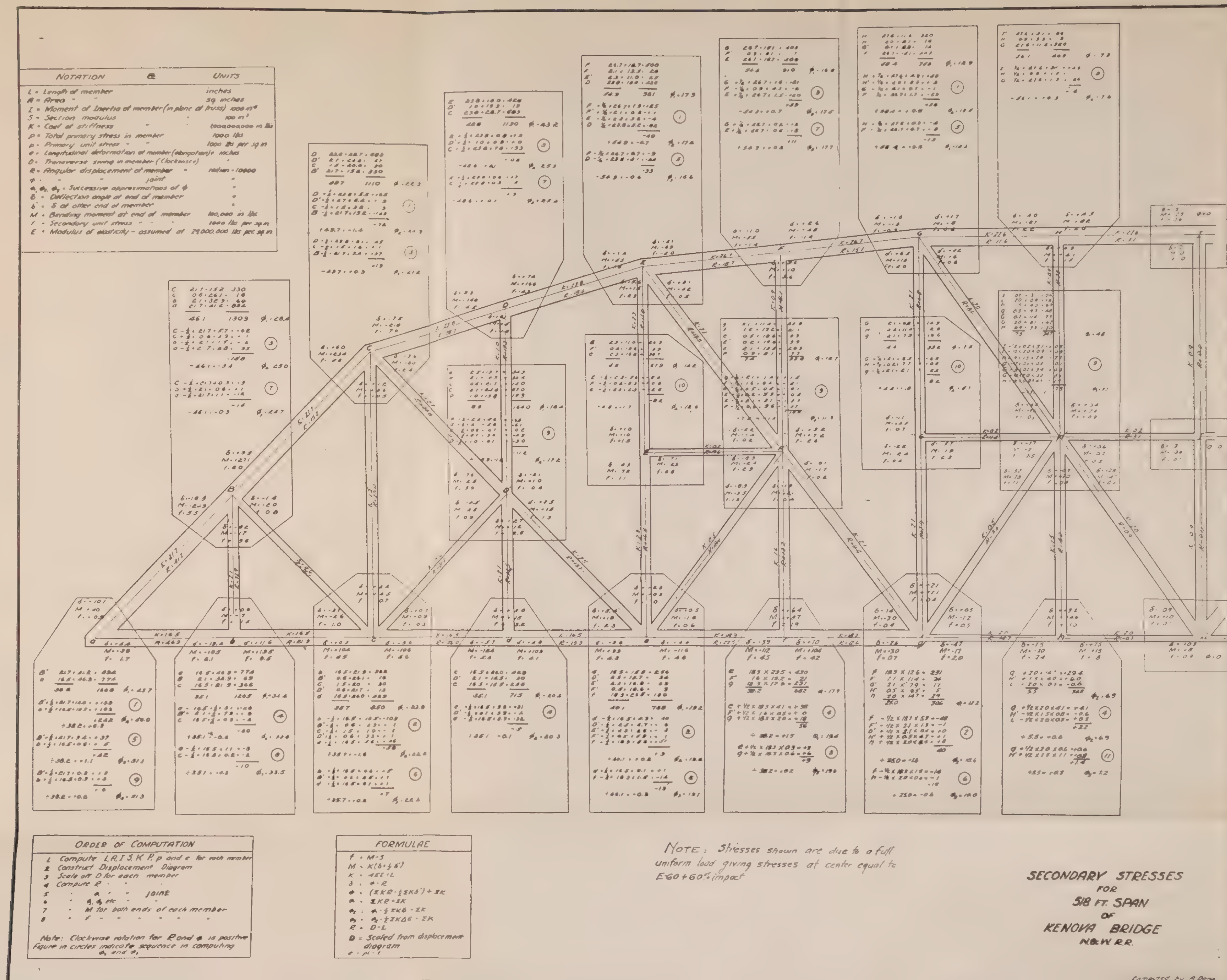


FIG. 177a

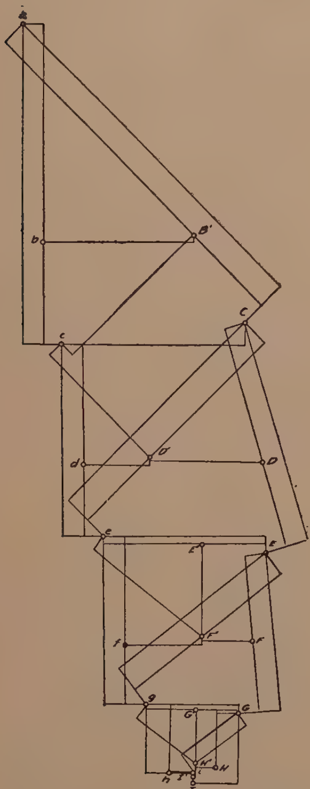
(To face page 330)

and the "twist" at the joint in question and not at all on the neighboring joint twists, i.e., that

$$M_{mn} = \frac{4EI}{L}(\phi_m - R), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

rather than

$$M_{mn} = \frac{2EI}{L}(2\phi_m + \phi_n - 3R). \quad . \quad . \quad . \quad . \quad . \quad (b)$$



DISPLACEMENT DIAGRAM

FIG. 177*b*

TABLE A

TABULATION OF VALUES FOR EACH MEMBER											
	L	A	I	S	K	P	p	e	D	R	f
aB	553	423	1034	45/40	21.7	-4830	-11.4	-.218	2.28	41.2	5.5
B'c	"	"	"	"	"	-4520	-10.7	-.204	0.84	15.2	6.0
CD	405	321	830	37/31	23.8	-4370	-13.6	-.190	1.16	28.7	7.0
DE	"	"	"	"	"	"	"		0.73	18.0	4.3
EF	390	357	898	39/34	26.7	-4900	-13.7	-[illegible]	"	18.7	2.0
FG	"	"	"	"	"	"	"	"	0.59	15.1	1.2
GH	388	369	922	40/35	27.6	-5200	-14.1	-.189	0.54	11.6	2.2
HI	"	"	"	"	"	"	"	"	0.12	3.1	2.0
ab	"	258	55.3	23	16.5	+3390	+13.1	+.175	1.82	46.9	8.1
bc	"	"	"	"	"	"	"	"	0.85	21.9	8.5
cd	"	"	"	"	"	"	"	"	1.01	26.0	5.4
de	"	"	"	"	"	"	"	"	0.60	15.5	4.3
ef	"	315	61.1	25	18.3	+4370	+13.9	+.186	0.41	23.5	4.6
fg		"	"	"	"	"	"	"	0.49	12.6	4.2
gh	"	320	6.8	8.5	2.0	+5040	+5.7	+.210	0.57	14.7	2.4
hi	"	"	"	"	"	"	"	"	-0.01	-0.3	1.8
CD'	553	126	12.8	8.4	2.7	+1470	+11.7	+.223	1.37	24.8	3.0
D'e	"	115	12.0	7.9	2.5	+1150	+10.0	+.191	0.76	13.7	2.3
EF'	639	111	11.7	7.7	2.1	+1120	+10.1	+.222	0.86	13.5	0.5
F'g	"	"	"	"	"	+830	+7.5	+.165	0.73	11.4	0.4
GH'	670	107	11.4	7.5	2.0	+562	+5.3	+.122	0.54	8.1	0.8
H'i	"	"	"	"	"	+280	+2.6	+.060	0.06	0.9	[illegible]
Cc	786	108.4	103.7	6.80	1.5	+734	+6.8	+.184	1.57	20.0	0.7
EE'	508	82.6	99.5	6.54	2.3	-310	-3.8	-.066	0.56	11.0	2.3
E'e	"	"	"	"	"	"	"	"	0.85	16.8	1.1
GG'	546	"	"	"	2.1	-153	-1.9	-.036	0.37	6.8	2.0
G'g	"	"	"	"	"	"	"	"	0.43	7.9	0.4
B'b	393	58.7	7.12	4.70	2.1	+277	+4.7	+.064	1.29	32.9	3.6
D'd	"	"	"	"	"	"	"	"	0.57	14.5	2.2
F'f	508	"	"	"	1.6	"	"	+.082	0.67	13.2	1.9
H'h	546	"	"	"	1.5	"	"	+.089	0.22	4.0	1.0
B'c	553	45.9	2.97	2.65	0.6	-323	-7.0	-.134	1.44	26.1	1.0
D'c	"	"	"	"	"	"	"	"	1.20	21.7	0.9
F'e	639	"	"	"	0.5	-288	-6.3	-.139	1.14	18.6	1.3
H'g	670	"	"	"	"	-279	-6.1	-.141	0.63	9.5	1.1
DD'	508	37.7	4.26	2.80	1.0	-82	-2.2	-.038	0.98	19.3	1.8
FF'	546	"	"	"	0.9	"	"	-.041	0.44	8.1	3.6
HH'	"	"	"	"	"	"	"	"	0.18	3.3	1.5
II'	"	"	"	"	"	"	"	"	0	0	0
I'i	"	"	"	"	"	"	"	"	0	0	0
EF'	388	19.4	0.62	0.83	0.2	0	0	0	0.76	19.6	2.9
G'H'	"	"	"	"	"	0	0	0	0.44	11.4	2.5
HI	"	"	"	"	"	0	0	0	0.12	3.1	0.7

This can never be exactly true, but it serves as a very useful approximation in this case. $R = \frac{D}{L}$ being obtained from a Williot diagram in advance, it is seen that when equations of the type (a) are substituted for type (b) in $\Sigma M = 0$ for any joint, we get a set of *independent equations which require no simultaneous solution.*

(4) The calculation at G may be followed through as typical. The calculation for ϕ_1 is obvious. For ϕ_2

$$= \phi_1 - \tfrac{1}{2}\Sigma K\delta' \div \Sigma K = (\text{approx.})\phi_1 - \frac{\tfrac{1}{2}\Sigma K(\phi'_1 - R)}{\Sigma K},$$

the values of $\phi'_1 - R$ are written out for the adjacent joints—thus for H, $\phi'_1 = 7.3$ and for member GH, $R = 11.6$ and $K = 27.6$.

$$\therefore\ -\frac{K}{2}\delta' = -\frac{K_2}{2}(\phi'_1 - R) = -\tfrac{1}{2} \times 27.6(7.3 - 11.6) = +59.$$

The calculations for H', G' and F are precisely similar. All are shown under calculations marked ①.

For the third approximation, ϕ_3, it is evident from calculations ① that the effect of H' and G' is entirely negligible.

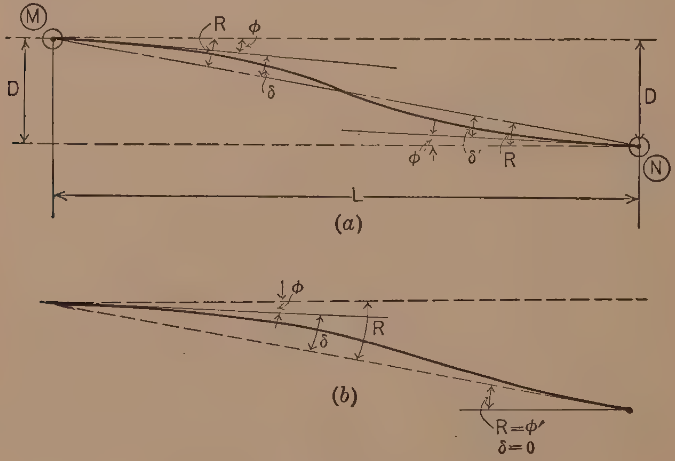


FIG. 178

R being a constant, we have

$$\delta' = \phi' - R,\ \Delta\delta' = \Delta\phi' = \phi_2 - \phi_1,$$

and considering the adjacent joint H

$$\Delta\delta' = \phi_{2H} - \phi_{1H} = 7.6 - 7.3 = +0.3, \text{ and}$$
$$-\tfrac{1}{2}K\Delta\delta'_H = -\tfrac{1}{2} \times 26.7 \times 0.3 = -4.0, \text{ and}$$
$$-\tfrac{1}{2}K\Delta\delta'_F = -\tfrac{1}{2} \times 26.7 \times 0.7 = -9.0.$$

(5) All computations marked ① are made prior to any marked ②. Therefore, the results of ① are available for use in calculations ② wher-

ever they apply. Thus, in computing ϕ_2 at joint c, only first approximate values of ϕ are available at b, B', D' and d, but at C the second approximation for ϕ has already been made, and hence δ' is computed as $\phi_2 - R$ rather than $\phi_1 - R$.

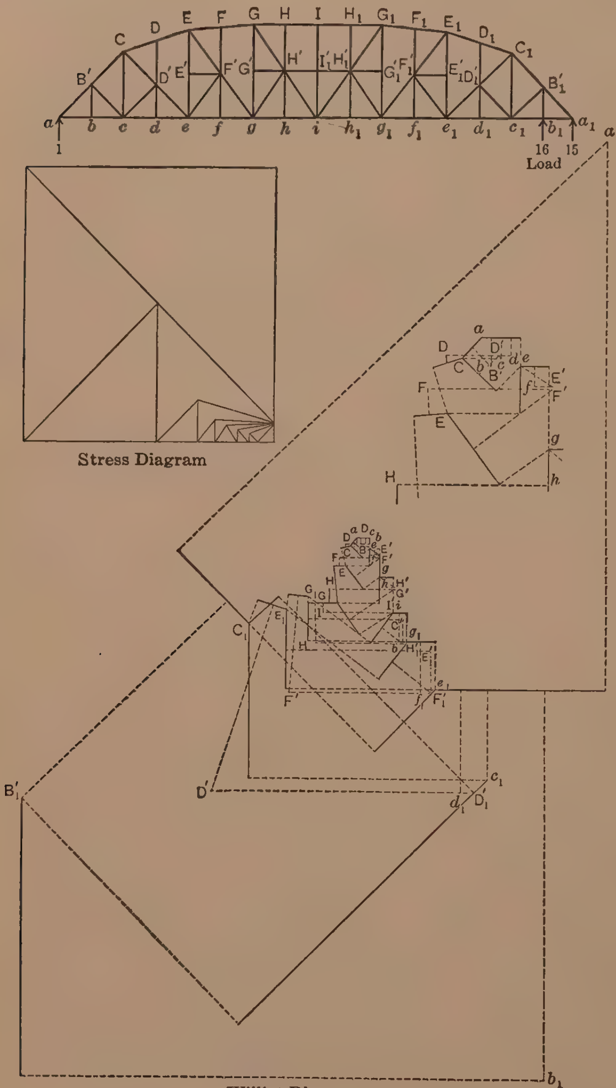
Calculations ① and ② have all been made before any calculations ③. Therefore, at B' in computing δ' for the adjacent joints a, c and C, the second approximation for ϕ is available and is used. At joint b, however, the second approximation has not been made, and hence ϕ_1 must be used.

Similar remarks apply to other calculations. This order is not obligatory, but serves to hasten the convergence somewhat. Outside the panel adjoining the center, where, from symmetry, it is known in advance that the values of ϕ must be small, the range in the ϕ's for chord points is from 10 at g to 51 at a. In any region of the size affecting a single joint, it will be noted that the variation in the joint twists is not great.

138. *Example* 3.—Table A shows the set-up of the 43 simultaneous equations for the same truss as example 2, but with a 1000# load at b. Fig. 179 shows the truss and loading and the Williot diagram. This and the table form a part of a set of calculations made by the authors for the purpose of constructing influence lines for the secondary stresses in the various members. The equations are solved by repeated trial by a method identical with that used in problem I, except that instead of taking the values of ϕ in a given group to be equal for a preliminary assumption, as was there done, the first value was taken as $\frac{\Sigma D}{\Sigma L}$, the summation to include all members entering a joint. The method of successive substitutions is at a decided disadvantage here as compared to the two previous problems. The character of the loading—a single concentration at the extreme end—is such as to produce sharp variations in ϕ for joints closely adjacent, hence a suitable first assumption is difficult. None the less, it will be noted that for all important values of ϕ, the second approximation is practically exact.

139. Remarks on Method of Solution.—The foregoing problems illustrate tolerably completely the solution of the secondary stress problem. In but one particular is this more difficult than other statically indeterminate problems previously treated, i.e., in the matter of the large number of equations to be solved simultaneously. In other respects, the theory is rather simpler than that of frames and arches. The matter of solving a large number of simultaneous equations, such as occur in any statically indeterminate problem of high degree, is generally regarded by the engineering profession as a rather staggering

task, and with considerable reason. It is quite true that the set of 43 equations of table A, example 3, can be solved in exactly the same manner as the 5 equations of example 1, table B, and if proper care and skill be exercised in selecting the order of elimination and proper vigilance maintained in avoiding errors, no special difficulty will be encountered, though the time required at best will be considerably more, and except in the case of highly skilled and experienced computors, will probably be several times as great. The experience of many competent designing engineers indicates that difficulties in the order of elimination and in the accumulation of errors are of common occurrence. The previous discussion has therefore laid special emphasis on methods of solution which avoid this laborious process. Opinions of authorities differ on the advantages of solution by successive substitution, but the authors believe that the solution of the



Stress Diagram

Williot Diagram

Fig. 179

TABLE A

	ϕ_a	$\phi_{B'}$	ϕ_b	ϕ_C	ϕ_c	ϕ_D	$\phi_{D'}$	ϕ_d	ϕ_E	$\phi_{E'}$	ϕ_e	ϕ_F	$\phi_{F'}$	ϕ_f	ϕ_G	$\phi_{G'}$
a	812.2	259.4	146.7													
B'	259.4	1084.2	18.0	259.4	5.3											
[illegible]	146.7	18.0	622.8		146.7											
[illegible]		[illegible]		1008.6	13.2	208.0	23.7									
C		5.3	146.7	13.2	654.4		5.3	156.7								
D				208.0		848.8	8.4		208.0							
[illegible]				23.7	5.3	8.4	155.0	18.0			22.1					
[illegible]					156.7		18.0	662.8			156.7					
E						208.0			936.2	19.2		223.0	17.9			
E'									19.2	80.0	19.2		1.6			
e							22.1	156.7		19.2	728.4		4.6	161.6		
F									223.0			908.0	8.0		223.0	
F'									17.9	1.6	4.6	8.0	127.8	13.9		
f											161.6		13.9	674.2		
G												223.0			1012.8	17.9
G'															17.9	74.8
g													17.9	161.6		17.9
[illegible]															249.0	
H'															16.5	1.6
[illegible]																
[illegible]																
[illegible]																
i																
H_1																
H_1'																
h_1																
G_1																
G_1'																
g_1																
F_1																
F_1'																
f_1																
E_1																
E_1'																
e_1																
D_1'																
D_1																
d_1																
C_1																
c_1																
[illegible]																
[illegible]																
[illegible]																

	ϕ_g	ϕ_H	$\phi_{H'}$	ϕ_h	ϕ_I	$\phi_{I'}$	ϕ_i	ϕ_{H_1}	$\phi_{H_1'}$	ϕ_{h_1}	ϕ_{G_1}	$\phi_{G_1'}$	ϕ_{g_1}	ϕ_{F_1}	$\phi_{F_1'}$	ϕ_{f_1}
a																
B'																
[illegible]																
[illegible]																
C																
D																
[illegible]																
[illegible]																
E																
E'																
e																
F																
F'	17.9															
f	161.6															
G		249.0	16.5													
G'	17.9		1.6													
g	438.8		4.4	17.6												
[illegible]		1012.0	8.0		249.0											
H'	4.4	8.0	123.0	12.9		1.6	16.5									
[illegible]	17.6		12.9	36.2			[illegible]									
[illegible]		249.0			1012.0	8.0		249.0								
[illegible]			1.6		8.0	384.0	8.0		1.6							
i			16.5	17.6		8.0	152.4		16.5	17.6						
H_1					245.0			1012.0	8.0		249.0					
H_1'						1.6	16.5	8.0	123.0	12.9	16.5	1.6	4.4			
h_1							17.6		12.9	96.2			17.6			
G_1								249.0	16.5		1012.8	17.9		223.0		
G_1'									1.6		17.9	74.8	17.9			
g_1									4.4	17.6		17.9	438.8		17.9	161.6
F_1											223.0			908.0	8.0	
F_1'													17.9	8.0	127.8	13.9
f_1													161.6		13.9	674.2
E_1														223.0	17.9	
E_1'															1.6	
e_1															4.6	161.6
D_1'																
D_1																
d_1																
C_1																
c_1																
[illegible]																
[illegible]																
[illegible]																

	ϕ_{E_1}	$\phi_{E_1'}$	ϕ_{e_1}	ϕ_{D_1}	$\phi_{D_1'}$	ϕ_{d_1}	ϕ_{C_1}	ϕ_{c_1}	ϕ_{B_1}	ϕ_{b_1}	ϕ_{a_1}	$\frac{\Sigma ID}{L}$
a												43.77
B'												44.22
[illegible]												[illegible]
[illegible]												[illegible]
C												24.00
D												43.46
[illegible]												[illegible]
[illegible]												[illegible]
E												38.26
E'												1.87
e												32.76
F												58.30
F'												3.67
f												39.04
G												54.58
G'												1.19
g												28.17
[illegible]												[illegible]
H'												2.82
[illegible]												[illegible]
[illegible]												[illegible]
[illegible]												[illegible]
i												10.42
H_1												37.41
H_1'												0.99
h_1												5.64
G_1												66.75
G_1'												−1.39
g_1												40.44
F_1	223.0											41.01
F_1'	17.9	1.6	4.6									−1.76
f_1			161.6									44.91
E_1	936.2	19.2		208.0								75.30
E_1'	19.2	80.0	19.2									−4.0[illegible]
e_1		19.2	728.4		22.1	156.7						69.00
D_1'	208.0			848.8	8.4		208.0					27.99
D_1			22.1	8.4	155.0	18.0	23.7	5.3				−9.68
d_1			156.7		18.0	662.8		156.7				57.98
C_1				208.0	23.7		1008.6	13.2	259.4			73.05
c_1					5.3	156.7	13.2	654.4	5.3	146.7		142.08
[illegible]							259.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]								146.7	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]									[illegible]	[illegible]	[illegible]	[illegible]

	ϕ_a	$\phi_{B'}$	ϕ_b	ϕ_C	ϕ_c	ϕ_D	$\phi_{D'}$	ϕ_d	ϕ_E	$\phi_{E'}$	ϕ_e	ϕ_F	$\phi_{F'}$	ϕ_f	ϕ_G	$\phi_{G'}$	ϕ_g	ϕ_H	$\phi_{H'}$	ϕ_h	ϕ_I	$\phi_{I'}$
ΣD	39	33	26.5	21	19.8	28.5	23	21.3	24.5	16	24	34	24	30.5	32	11	31.5	24	18.5	34	44	0
ΣL	[illegible]	1105	777	957.5	777	810	1105	777	795	[illegible]	777	780	1279	777	778.5	1092	777	777	1340	777	777	1092
$\frac{\Sigma D}{\Sigma L}$	[illegible]	.0298	.0341	.0219	.0255	.0352	.0208	.0274	.0308	[illegible]	.0309	.0436	.0188	.0393	.041	.010	.0406	.0309	.0138	.0438	.0566	0
	.0396	.0263	.0309	.0178	.0224	.0382	.0170	.0274	.0222	.0085	.0294	.0464	.0104	.0419	.0365	.0040	.0468	.0199	.0017	.0390	.0698	−.0102
	.0399	.0264	.0309	.0175	.0227	.0412	.0177	.0274	.0210	.0107	.0292	.0497	.0106	.0396	.0386	.0040	.0468	.0180	.0037	.0396	.0745	−.0141
	.0399	.0264	.0308	.0168	.0225	.0416	.0176	.0274	.0195	.0111	.0292	.0495	.0106	.0394	.0386	.0044	.0477	.0163	.0033	.0393	.0760	−.0151
																		.0160			.0772	

	ϕ_i	ϕ_{H_1}	$\phi_{H_1'}$	ϕ_{h_1}	ϕ_{G_1}	$\phi_{G_1'}$	ϕ_{g_1}	ϕ_{F_1}	$\phi_{F_1'}$	ϕ_{f_1}	ϕ_{E_1}	$\phi_{E_1'}$	ϕ_{e_1}	ϕ_{D_1}	$\phi_{D_1'}$	ϕ_{d_1}	ϕ_{C_1}	ϕ_{c_1}	ϕ_{B_1}	ϕ_{b_1}	ϕ_{a_1}	
ΣD	44	15	4	32	37	−14	36	25	−15	35	57	−35	51	21	−47	47	68	131	−210	−236	−699	
ΣL	777	777	1340	777	778	1092	777	780	1279	777	795	1016	777	810	1105	777	957.5	777	[illegible]	[illegible]	[illegible]	
$\frac{\Sigma D}{\Sigma L}$	.0566	.0193	.003	.041	.047	−.0128	.0464	.0321	−.0117	.045	.0717	−.0344	.065	.0259	−.042	.0605	.0710	.168	[illegible]	[illegible]	[illegible]	[illegible]
	.0566	.0113	−.0123	.0393	.0538	−.0410	.0749	.0160	−.0392	.0400	.0680	−.0834	.0740	−.0032	−.0968	.0333	.1065	.1631	−.1362	−.330	−.974	1st approxim'n
	.0610	.0066	−.0136	.0362	.0605	−.0491	.0775	.0156	−.0408	.0317	.0798	−.0840	.0841	−.0089	−.0986	.0339	.1082	.2830	−.0889	−.2780	−.999	2nd approxim'n
	.0612	.0037	−.0143	.0352	.0619	−.0510	.0840	.0156	−.0413	.0289	.0790	−.0891	.0858	−.0121	−.1020	.0033	.0968	.2710	−.0840	−.302	−1.010	3rd approxim'n
		.0041												−.0091	−.0986	.0057	.0954	.2850	−.0888	−.296	−1.005	4th approxim'n

secondary stress problem as it arises in office design can always be made much easier and more quickly by such methods as illustrated in examples 1 and 2 than by attempting the ordinary standard method of elimination.*

140. *Approximate Methods.*—Where it is desired to investigate the secondary stresses at a single joint or a few joints, it will generally be satisfactory to limit the equations to one joint beyond the group in question, and solve this set. For example, if it is desired in a truss such as that of Fig. 176 to investigate a, B and C only, good results may be obtained by assuming $\phi = \frac{O}{L} = 0$ for all joints except a, B', C, b, c, D and D' and carrying through the solution for these 8 values. If the stress at B' only is desired, only a, b, c, C and B' need be used.

141. Influence Lines.—The construction of influence lines for secondary stresses is very laborious, since a complete solution must be made for the case of a unit load at each joint. It will seldom be necessary in practice to make a solution on this basis, but a good deal of useful general information may be gained from an influence line study of typical trusses. Figs. 180 and 181 show some representative influence lines for the Kenova span. From a designing standpoint, two points of importance may be noted.

(a) The effect of local concentration is relatively much greater on the secondaries than on the primaries in a truss of the Kenova type. On this account, the calculation of the former for an equivalent uniform load suitable for the primary stresses is likely to give much smaller values than would the actual concentrations.

(b) Most secondary stresses in chord members are a maximum under full loading, but important exceptions occur, as will be observed from the influence line for EFG at F. A loading of approximately $\frac{3}{4}$ of the span will for this case give a maximum total stress—primary plus secondary.

142. Trustworthiness of Secondary Stress Calculations.—This point is discussed briefly as a part of the whole subject of the dependability of statically indeterminate stress analysis in Chapter VIII. Little can be added here, but it may be of some interest to quote the following conclusions from the authors' analytical and experimental study of the stresses in the Kenova bridge.†

* Fuller discussion of this point will be found in a paper by D. B. Steinman, Transactions Am. Soc. C. E., Vol. LXVII, and in University of Minnesota Studies in Engineering No. 4, "Secondary Stresses in Kenova Bridge," pages 15–21.

† Maney and Parcel. University of Minnesota, Studies in Engineering No. 4, "Investigation of Secondary Stresses in the Kenova Bridge," page 3.

"An analysis of the experimental data would appear to justify the following conclusions:

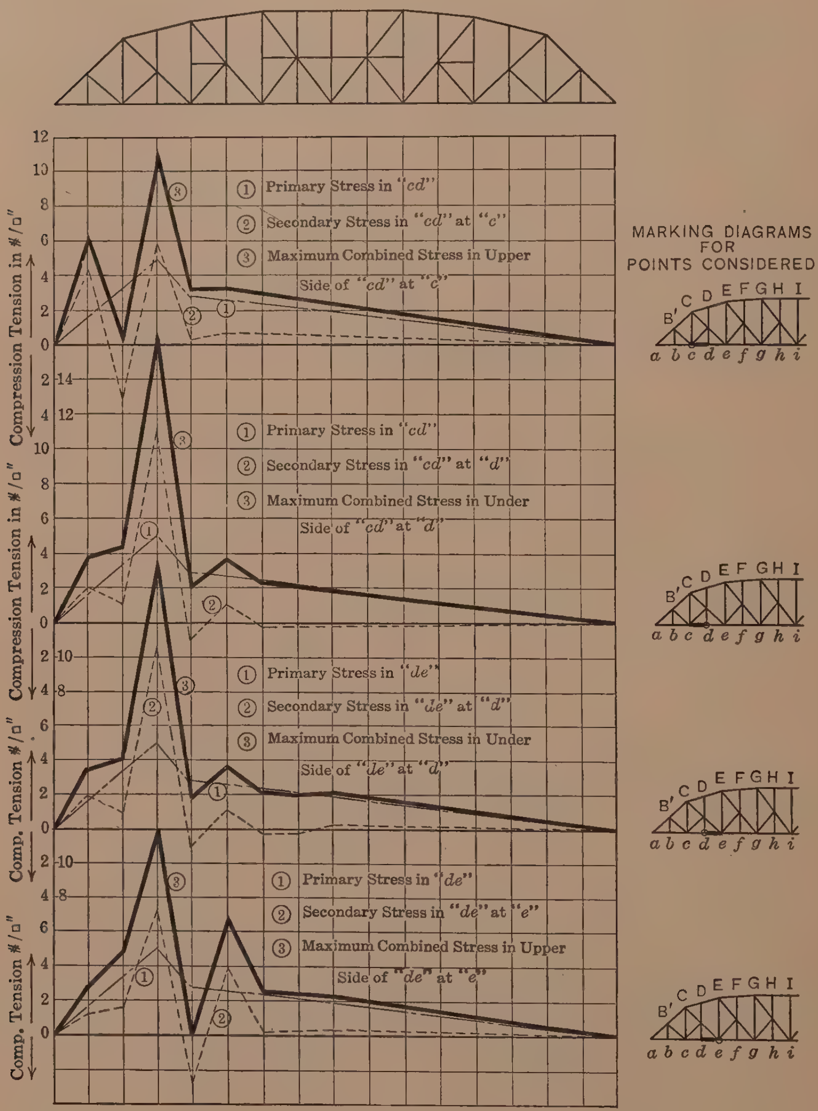


FIG. 180

"1. In all members of a truss of the Kenova type the actual secondary stresses may be expected to fall appreciably below the computed values,

due, it is believed, to the relieving effect of local induced deformations which are not considered in the conventional method of computation.

"2. For most members the secondary stresses computed by the con-

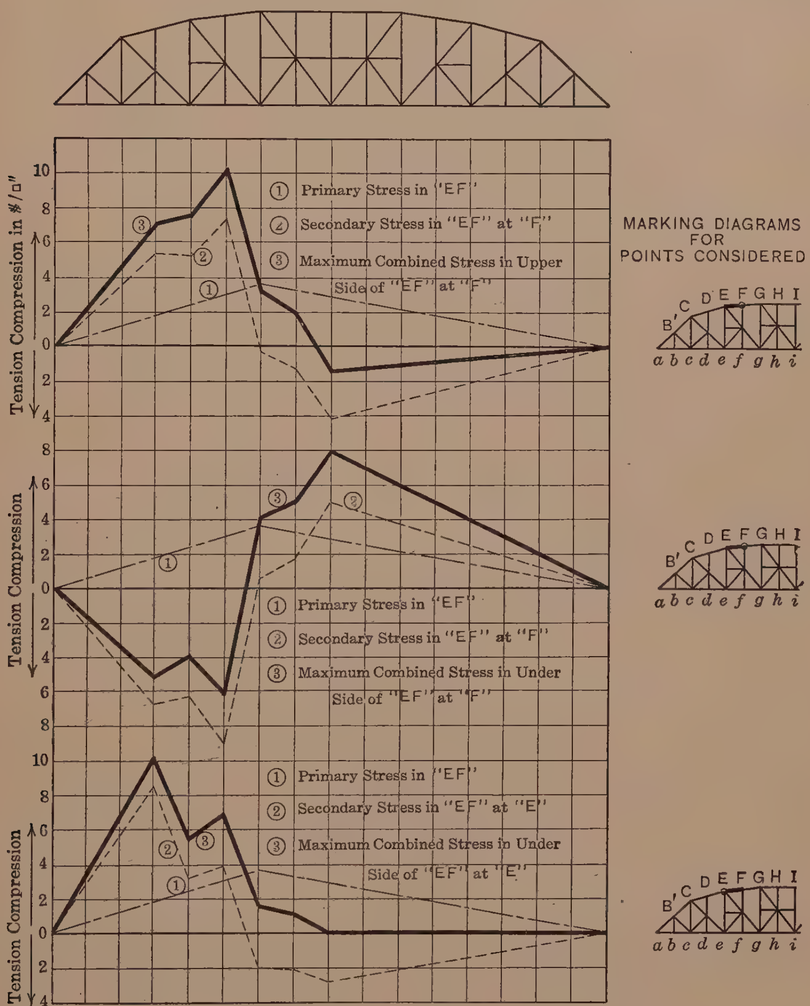


Fig. 181

ventional method, *using the same equivalent uniform load as for primary stresses*, are fair approximate values of the *maximum upper limit* of the actual secondaries.

"3. For some members the relieving effect is so great that the ordi-

nary calculation, if made for maximum conditions, gives altogether excessive results.

"4. The relieving factors appear to be amenable to rational analysis, and when properly taken account of in the calculations, give results in close agreement with the values actually observed. That is to say that the theory of truss action in the broad sense is verified, though it may in some cases be necessary to include factors ordinarily neglected in the conventional computations."

143. *Importance of Secondary Stresses in Bridge Design.*—Recent tendency in bridge design seems to be in the direction of a wider use of the solid riveted truss for spans of all lengths, and it may be said that the standard practice for pin-connected trusses is to build the top chord continuous. It is to be observed also that long simple spans of the Petit type have become relatively common, and the limiting length is constantly increasing. The importance of studying carefully the secondary stresses in long span (fully or partially) riveted trusses of the Petit type is generally admitted; indeed it may be doubted if a satisfactory design can be made otherwise.

It appears, therefore, that we can no longer regard the problem of secondary stresses as important only in relation to rare monumental structures, such for example as the Quebec bridge; the problem has become one of much more common occurrence, and a knowledge of secondary stress analysis has as important a place in the equipment of an expert structural engineer as the analysis of swing bridges, arches or frames.

CHAPTER VIII

GENERAL DISCUSSION OF STATICALLY INDETERMINATE CONSTRUCTION—HISTORICAL REVIEW—BIBLIOGRAPHY

A. General Discussion

144. Preliminary.—The statically indeterminate structure has never found favor generally with American engineers; as a matter of fact until quite recent times the attitude of the profession has been distinctly hostile to it. However, the increase in the number of monumental structures, to many of which statically indeterminate types are especially suited, and for which more careful analyses are required, the widening use of riveted construction, and more than anything else, perhaps, the remarkable development in the use of reinforced concrete (an essentially statically indeterminate type of construction in most cases) have, along with other causes, effected a very considerable change in the professional attitude, with the result that indeterminate construction is now much better understood and more widely used and, for suitable conditions, has many advocates. None the less, a sharp division of opinion remains, and it is probably no exaggeration to say that the majority of structural engineers in America still oppose statically indeterminate types wherever they can be avoided, and where their use is practically unavoidable they have rather limited confidence in the exact methods of analysis that have been proposed, preferring in many cases crude estimates based on judgment and experience.

Under such conditions it would seem not out of place to introduce the student, at least superficially, to some of the major points that are raised regarding the economy and reliability of indeterminate structures, the difficulties and uncertainties of the calculations and similar questions. It is the purpose of the present chapter to do this. To discuss the various questions thoroughly would require an independent treatise; only the very briefest outline can be attempted here. Also, as later noted, most of the questions raised cannot be answered conclusively in the present state to our knowledge; some of them may always remain, to an extent, a matter of opinion. The most that can be hoped from the short discussion presented here is to give the student the setting of the

subject and direct him to the more important sources of information. A brief historical review and bibliography are also added.

145. Review of Definition and Classification.—Before proceeding with the discussion it will be well to consider somewhat more critically the usual definitions and classification.

A statically determinate structure, as ordinarily defined, is one in which the reactions and internal stresses are fixed by the bare requirements of equilibrium, and are not at all conditioned by the cross-section make-up of the various parts of the structure, nor the elastic properties of the material, so long as these are such as to result in a strained structure of sensibly the same form as the original.*

The question may properly be raised here as to whether "internal stress" is to be taken as referring to *resultant* axial stresses and moment couples, or to stresses *on individual fibers.* If the latter interpretation, which is the strictly correct one, is used, then no structure of any magnitude is even approximately statically determinate as regards internal stresses. A structure simply supported and composed of "ideal" bars (perfectly straight, absolutely homogeneous, etc.), ideally jointed (frictionless pins exactly centered), is the type ordinarily visualized as the "perfect" statically determined structure, since for such members it appears fair to assume a uniform stress distribution regardless of the material or cross-section. This, however, ignores the fact that (1) the local distribution of stress at the pin bearing is not statically determined and (2) that in some or all bars there will be flexural stresses due to weight of members. Though emphasis is seldom placed on the point, it is obvious that these latter stresses are *always* statically indeterminate. The beam formula, $s = \frac{Mc}{I}$ is based on the assumption of a linear relation between stress and strain and the integrity of plane sections. These are propositions of experimental elasticity, not of statics. Neither is strictly true for any material, and the degree of approximation varies widely with the character of the material and the conditions of loading. The problem of the resistance of a beam to loads was, as a matter of fact, the problem from which the whole modern theory of elasticity developed.†

It is clear, then, that if a strict construction is put upon the definition, *no actual engineering structure is statically determinate.*

* Since this is never more than approximately true, every elastic structure, even with ideally perfect members and connections, is statically indeterminate. In all ordinary cases the elastic distortion is so slight that this point has only theoretical interest. See Chapter I, p. 12.

† See Historical Review, Art. 156*a*.

If we consider resultant stresses and moments only, the field of application of the definition is greatly broadened, but even so, few if any practical structures will conform to it. "Frictionless" joints have only an ideal existence; and even in a fully pin-connected bridge truss for example, the effect of the floor system and the lateral, portal and sway bracing is such as always to set up very appreciable statically indeterminate stresses. These are usually small, though in some cases they may reach considerable magnitude and importance. Ordinarily they do not explicitly affect the design, though they are among the many factors that lead to the adoption of a wide margin of safety and hence indirectly are taken account of.

146. Conventional Character of Classification.—If the preceding statements are correct, we must regard the grouping of structures into statically determinate and indeterminate as subject to a conventional rather than a strictly logical interpretation, if we are to give any significance to the classification. Even with such an understanding, there is considerable difficulty in drawing the line between the two groups. For example, a long-span fixed arch of reinforced concrete, carrying light highway traffic, may show a pressure line which follows so closely that of the corresponding 3-hinged type as to modify the extreme fiber stresses less than 25 per cent.

On the contrary, an extremely massive simple-span riveted truss with sub-panels may show secondary stresses as high as 60 per cent to 80 per cent of the corresponding primary stress.

Now, a fixed arch is unquestionably a triply indeterminate structure, while a simply supported truss, even with rigid joints, is to a very close approximation* statically determinate as to its *primary* (axial) stresses. But it would seem stretching logical classification to the limit to call the arch indeterminate and the riveted truss determinate.

There is no definitely established precedent to follow in such cases but generally speaking the usual practice is to class any structure as statically indeterminate in which the reactions, *principal* stresses or *primary* bending moments cannot be determined statically, and to treat as determinate any framework (used in the widest sense to include a single isolated beam, strut or tie) in which such stresses amd moments can be determined statically by ignoring so called "secondary" effects, these latter ordinarily including the effect of floor system and bracing, actual continuity at joints assumed as hinged, and such like.

147. Statically Indeterminate Structures in the Limited Sense.—The preceding definition would limit the use of the term *indeterminate structure* to structures with redundant supports, trusses with redundant

* See Chapter VII, page 319.

members and certain types of rigid frames. The latter type in nearly all cases is indeterminate of practical necessity; that is to say, there is no alternative simple type of rigid frame practically feasible. This is not true of the other two types; they usually have close analogues that are statically determinate. Thus, for example, we get most of the unique advantages of continuous construction in the cantilever; the advantages of arch action are furnished by the three-hinged as well as by the two-hinged or hingeless arch; the advantages of short floor panels combined with favorable diagonal inclinations are secured by the statically determinate sub-paneled truss of the Baltimore or Petit type as well as by the indeterminate Whipple truss. Most of the discussion regarding the relative advantages and disadvantages of statically indeterminate construction in the past has referred to structures of this character, hence for convenience of treatment we shall in the following discussion consider the term "indeterminate structure" to apply primarily to such frameworks except as otherwise indicated.

148. Merits and Defects of Statically Indeterminate Construction.—As noted in Art. 144, the question of the relative merits of statically determinate and indeterminate construction has given rise to a sharp difference of opinion among professional engineers, and has been the subject of much controversy. Only a brief and inadequate account of the matter can be given here, but it is hoped that it will at least make clear some of the main points at issue, the difficulties involved in settling them, and direct the attention of the student to some of the original researches in the field.

It will be convenient to treat the subject under the three heads of (1) relative economy of material in indeterminate structures, (2) their reliability under service conditions, and (3) the trustworthiness of the methods of analysis and the difficulty of making the design calculation.

149. *Economy of Material.*—In spite of the many studies that have been made and published, there still remains a wide diversity of professional opinion as to whether or not a statically indeterminate type of construction is likely to show an economy of material as compared to an alternative type of simple structure. Structural design is not as yet an exact science by any means; there is no unanimity even among leading authorities in the matter of provision for impact, temperature changes, reversal of stress, possible inaccuracies in fabrication, settlement of supports and like matters. Details are likely to reflect more or less the personal opinions—perhaps prejudices—of the designer. Under such conditions it is not strange that competent investigators arrive at very different conclusions.

We may note some results of specific studies in comparative economy

of determinate and indeterminate types. Since the arch has been the subject of more investigation than perhaps any other type, it will serve very well as an example.

(*a*) Merriman and Jacoby* investigated the result of placing a crown hinge in the 550-ft. two-hinged spandrel braced arch bridge over the Niagara river and found that if the hinge were placed in the upper chord the weight (arches alone) would be decreased 11.8 per cent.

(*b*) Prof. C. W. Hudson made a comparative study † of the weight of a 200-ft. span highway bridge designed as a three-hinged and two-hinged arch. The latter structure showed an economy (main material and details both considered) of $5\frac{1}{2}$ per cent.

(*c*) Dr. J. A. L. Waddell has made a very elaborate study of the "Economics of Steel Arch Bridges."‡ Among the conclusions he arrives at are that for railway bridges a 500-ft. braced-arch rib will require about 5 per cent less metal when designed as a two-hinged arch than as a three-hinged arch, while a hingeless arch will require about 5 per cent more metal than the latter. For highway bridges the three-hinged type is about 8 per cent heavier than the two-hinged and 2 per cent lighter than the hingeless type. This includes weights of both main members and details.

(*d*) Prof. M. A. Howe § made a comparative study of a 416-ft. arch truss, finding that a three-hinged design was 30 per cent heavier than a hingeless type and about 8 per cent heavier than one with two hinges.

(*e*) Prof. W. Dietz of Munich states¶ that a very careful comparison of material required for a two-hinged and three-hinged arch for the 110-ft. span of the Hacker bridge in Munich, indicated that 11.3 per cent excess was required for the latter type.

The inconclusiveness of these results when taken as a whole needs no comment, though it is but fair to say that a considerable part of the discrepancy would be explained if specifications in each case were analyzed.

* See "Roofs and Bridges," Part IV, pp. 282-4.

† "Comparison of Weights of the three-hinged and a two-hinged Spandrel-braced Parabolic Arch"—Trans. A. S. C. E., Vol. XLIV, pp. 20-30. This comparison was made for the crown hinge in the lower chord in the three-hinged arch—a location economically unfavorable.

‡ Trans. A. S. C. E., Vol. LXXXIII, pp. 1-41. This paper, including the discussion, forms the most up-to-date and complete treatment of the subject that has appeared. It covers a much wider field than the comparison of determinate and indeterminate types.

§ A note by Dr. Waddell, loc. cit. page 19.

¶ Discussion of paper by Frank H. Cilley—"The Exact Design of Statically Indeterminate Frameworks. An Exposition of its Possibility but Futility." Trans. A. S. C. E., Vol. XLIII, page 424.

When we pass from the results of specific studies to general statements, we find them equally conflicting. Johnson, Bryan and Turneaure* state that the hingeless arch is "slightly more economical than the hinged types"; C. B. McCullough † says the hingeless arch "probably requires less metal than the three-hinged type"; though "no definite relation has ever been established"; F. C. Kunz ‡ considers that on the average the three-hinged arch will be about 15 per cent lighter than either the two-hinged or hingeless types, and J. W. Balet§ states that for suitable crossings the two-hinged and hingeless types will be more economical than any others that can be found.

These results relate to arches only but they are fairly representative of the whole field of indeterminate construction.

150. An ambitious and brilliant attempt to deal with the whole question of relative economy of indeterminate frame works by general mathematical reasoning has been made by F. H. Cilley.¶ Mr. Cilley supports the thesis that for every statically indeterminate type it is at least *theoretically* possible to devise a determinate type with approximately the same figure of inclusion which will carry the same loads with less material. He therefore concludes that, in general, structural redundancy means structural waste.

The method of investigation followed involves an enormous amount of detail for any but the simplest cases, and Mr. Cilley therefore limited his specific studies to a few ideal jointed frames of a rather primitive type. That he has made out a very convincing case for the abstract proposition as stated in the preceding paragraph must be conceded. This has served to dispose once for all (if there were need to do so) of the notion that there is any intrinsic economic advantage in statically indeterminate construction.

Beyond this, the investigation can hardly be said to have settled the economic question, for two reasons. (*a*) Few engineers are willing to accept as conclusive any economic comparison of two types of structures which does not involve a fairly complete design of each and a study of all important details, and (*b*) even Mr. Cilley himself did not claim that the alternative types of construction by means of which he transformed the indeterminate types into determinate types of less

* "Modern Framed Structures," Part II, page 185.

† Hool and Kinne, "Movable and Long Span Bridges," page 362.

‡ Quoted by J. A. L. Waddell, loc. cit. page 19.

§ "Analysis of the Elastic Arch," pp. 51, 100, 219.

¶ "The Exact Analysis of Statically Indeterminate Frameworks—an Exposition of its Possibility but Futility." Trans. A. S. C. E., Vol. XLIII, pp. 353-407. This paper and its discussion constitute the best general treatment of the subject that has so far appeared. They will well repay a careful study.

weight were always practically feasible; as was clearly brought out in the discussion, they might sometimes be quite impracticable.*

One point of general interest may be noted here. As the student has learned from the preceding chapters, the design of any statically indeterminate structure is a matter of repeated trial. A section is assumed for each member, the stresses computed and the unit stress noted; if the latter is not in agreement with the specified value, the section is revised and the whole process repeated, using the new value, and so on until a satisfactory agreement is reached.

For many cases a few approximations lead to an agreement which for all practical ends is exact. There can, however, be no general assurance on this point. We have noted in Chapter IV that for the bracing in the short middle panel of a three-span swing bridge, the unit stress remains nearly the same regardless of the variation in the cross-section of the members. Johnson, Bryan and Turneaure † note that a similar difficulty arises in the design of a hingeless-arch truss of the spandrel-braced type. Indeed, one of the clearest examples of the point in question is the ordinary beam with solid web. We may fix the extreme fiber stress, but the stress on any other fiber is *fixed by the elastic distortion of the beam*, and cannot be made to agree with a prescribed value. Though, as just noted, it may be of no real importance in many cases, this is an inherent defect of statically indeterminate construction so far as economy of material goes.

151. We may sum up the question of economic advantage conservatively by saying that, other things being equal, there is no good reason to expect that a statically indeterminate framework will show any economy in material over a determinate type; that it many cases it will be at a clear disadvantage in this regard, but that in other cases especially favorable to it, it will be more economical of material than any other type *practically feasible.*

It should be noted again that we are here considering economy of *material* only; this does not always coincide with total economy.

152. *Reliability.*—Since in any statically indeterminate structure a member cannot change length nor a support shift its relative position without setting up stresses throughout the structure, it is clear that the effects of inaccuracies in the lengths of members, of changes in temperature and settlement of foundations require very careful consideration. There can be no doubt that its sensitiveness to such effects constitutes a valid general criticism against indeterminate construction and it

* See particularly Professor Ritter's discussion on page 419, of the Trans. A. S. C. E., previously cited.

† "Modern Framed Structures," Part II, page 185.

has been a large factor in preventing a wider use of such construction—justly so in many cases. But it must not be forgotten that the importance of these effects varies widely with different conditions. With rigid inspection and improved modern methods of fabrication, inaccuracies in the fit of the members can no longer be regarded as serious. Temperature of course differs greatly in different localities, and the effect of temperature varies widely with different kinds of structures. In an arch a uniform change of temperature will stress the entire structure; in flat arches this effect may become very great—50 per cent or more, of the maximum stresses due to loading. On the other hand for arches with large rise and shallow rib the maximum temperature stresses may be less than 20 per cent of the full load and hence according to most specifications negligible.

In a continuous truss, a uniform change of temperature throughout the structure produces no stress, and generally the effect of unequal changes in different parts may be safely ignored.

The practical importance of settlement of supports likewise varies greatly with the individual structure. It is most important to note that in the design of foundations it is customary to adhere to a specified *unit* soil pressure, whether the pier or abutment be large or small, and since it is the unit pressure and not total pressure which governs the yield, there is no reason to expect this to be larger for a large span than for a small one. Now it is clear that for a deep plate girder, continuous over two 50-ft. spans, a relative settlement of $1\frac{1}{2}$ in. in the middle support might be disastrous while for a truss spanning two 500-ft. openings, such a settlement would affect the stresses relatively little. Emphasis on this fact has brought a considerable weight of professional opinion to favor a more liberal use of continuous construction for long spans. More or less similar conclusions apply to large scale indeterminate structures of other types.

As regards reliability of behavior of statically indeterminate structures under service conditions, then, we reach the same general conclusions as noted in Art. 151 relative to economy. The type of construction has certain inherent disadvantages, in many cases these are so important as to rule it out altogether, while under other conditions they have little or no practical weight, so that if indeterminate construction is otherwise advantageous, it may be used with complete confidence.

153. *Validity of Methods of Analysis of Indeterminate Stresses.*—Strictly speaking it is manifestly impossible to analyze the stresses in any statically indeterminate structure to the same degree of accuracy as in the case of a statically determined structure. The redundant reactions and stresses depend not only upon the requirements of static equilibrium,

but also in general, upon the elastic deportment of the structure as a whole, and hence an additional source of uncertainty is always involved in their calculation.*

This point has frequently been emphasized as an important general defect of statically indeterminate structures, and a reason for avoiding their use where possible. Whether as a matter of fact the uncertainties in the elastic behavior of such structures are sufficient to invalidate seriously the theory as a guide to practical design is a question which can be settled only by experimental investigation. No comprehensive investigation of the subject has ever been made, and a conclusive answer must await future researches.

There is available, however, a very considerable body of data, results of special tests, having a most important bearing on the question even if not entirely conclusive. Some of these results will be noted.

a. Tests on Steel Structures.

(1) Moore and Wilson† made laboratory tests to determine if the assumption of absolutely rigid joints in steel building frames was justifiable. They found that for the two most favorable types of connection the error due to slip was in one case from 1 per cent to 3 per cent and in the other from 2 per cent to 6.8 per cent.

This test was not a check test on indeterminate stresses in general, but inasmuch as the question of the actual rigidity of the joints in a stiff frame is fundamental, the results of the test are very significant.

(2) Horoi‡ made a test on a small model truss to check the calculated secondary stresses. Measurements were taken at three points, giving values of 600, 800 and 650 lbs. per sq. in. respectively. The corresponding computed values were 715, 830 and 750 lbs. per sq. in.

(3) Experimental investigation of secondary stresses has been conducted by a committee of the American Railway Engineering Association, and comparative results for a 105-ft. span pony Warren truss are given in Bulletin No. 163. There is a wide spread in the individual discrepancies, measured values apparently running under the computed in the chords and above in the web. The committee considers the agreement to be satisfactory considering all the possible sources of error in measurements.

(4) Maney and Parcel § made an experimental determination of the

*It is incorrect, however, to say that the indeterminate quantities can be computed no more accurately than the elastic deflections themselves. See pp. 12 and 267.

† University of Illinois Experiment Station Bulletin No. 104-1917.

‡ Journal of the College of Engineering, Imperial University of Tokio, November 30th, 1913.

§ See University of Minnesota, Studies in Engineering, No. 4, 1922.

secondary stress at four joints on the 518-ft. span Kenova bridge in 1917. The test was made under service conditions (no definite test load) and with instruments of no great refinement, so that quantitative comparison of measured and calculated stress are rather uncertain. Comparison with the conventionally computed stresses showed a wide range of error. When rigidity of floor and bracing and induced local distortion were taken into account the theoretical and measured stresses in nearly all cases agreed within 5 per cent to 20 per cent—a very satisfactory correspondence under the conditions.

(5) D. B. Steinman* has published a study of the dead load secondary stresses in the Hell Gate arch showing actual values far lower than those calculated by the conventional method. But these results have little bearing on the accuracy of the theory, since the type of joint used in this structure was such that the ordinary assumptions could not be expected to hold.

(6) The Swiss Technical Commission † has made a comprehensive experimental study of stresses in many types of structures, and have given especial attention to secondary stresses. The general conclusions of the Commission are that when all factors that can be conveniently included in the calculations are so included, the agreement is in general "satisfactory" and in many cases "very good." The results shown (graphically) in the reference here cited are in many cases in practically exact agreement, and nearly all are within an error of 15 per cent.

(7) During the erection of the Sciotoville bridge, opportunity was afforded (on account of erecting the structure under initial stress to reduce secondary stresses) to compare computed and measured deflections in a massive and complex framework. For practically all cases the error was but a few per cent; in many the agreement was exact. ‡

b. Tests of Concrete Structures.

(1) M. Abe§ has made laboratory tests to determine the applicability of the theory of statically indeterminate stresses to reinforced concrete frames. He concludes that the formulas will give stresses well within the limits of accuracy required for practical design.

(2) Slater and Richart¶ investigated experimentally the stresses in 2-legged rectangular reinforced concrete frames with different types of

* Trans. A. S. C. E., Vol. LXVII, 1914.

† See Schweiszerische Bauzeitung, Feb. 3, 1923. For further interesting information regarding this remarkable series of tests the authors are indebted to the kindness of M. M. Ros, Secretary to the Swiss Commission.

‡ See article by Clyde B. Pyle, Engineering News Record, Jan. 31, 1918.

§ University of Illinois Engineering Experiment Station Bulletin No. 107.

¶ Proc. A. C. I., Vol. XV, pp. 48-50 (paper by W. A. Slater).

brackets or haunches. The agreement between the test results and theory (as indicated in diagrams shown) was quite satisfactory.

(3) A very comprehensive analytical and experimental study of reinforced concrete flat slab structures has been made by Westergaard and Slater.* The authors conclude that when the results are reduced to a just basis for comparison the agreement between theory and experiment is fair.

Such problems as secondary stresses in steel bridges and the stresses in reinforced concrete frames puts the theory of indeterminate stresses to an especially severe test. The preceding results, even though falling short of complete verification will go far toward establishing confidence in the possibility of analyzing indeterminate stresses with sufficient accuracy for all practical needs. From the Sciotoville tests quoted it would seem altogether probable that a long span continuous truss (and by analogy, perhaps, a long span two-hinged arch or any similar type) can be analyzed with *practically* the same degree of exactness as a simple structure.

European engineers have in general, always accepted as trustworthy the standard methods of indeterminate stress analysis, and, as previously noted there appears to have been, in the last twenty years, a steady drift of opinion in this direction among American Engineers.

154. *Laboriousness of Calculations.*—Of the many inherent disadvantages of statically indeterminate construction it is possible that none has had more weight in influencing professional opinion than the fact that the analysis of the stresses is a very much more difficult and time-consuming task than in the case for a simple structure, yet, when considered rigidly on its merits, this objection, in general, has little to support it. The amount of time and expense involved in making the *stress calculations* for any structure of considerable magnitude is an exceedingly small item in the entire engineering of the structure. An expert computer will hardly require more than two or three days to make a complete analysis of the stresses in a moderate sized two-span continuous truss or a two-hinged arch, or any similar type. More complex problems, such as fixed arches, multiple rigid frames and secondary stress in riveted bridge trusses will ordinarily require considerably more time, but only under exceptional conditions will the statically indeterminate stress analysis require more than a week to ten days of the time of a trained expert.† Fairly accurate tentative analyses can be made by

* Moments and Stresses in Slabs. Proc. A. C. I., Vol. 17, 1921.

† These estimates are of course only crudely approximate, but they are based upon a considerable range of experience and observation, and are believed to be conservative; some computers report much greater rapidity in their calculations. It may be

approximate methods in a fraction of this time. If therefore, a definite saving of even 5 per cent or 6 per cent can be made either by the use of an alternate type of structure which is statically indeterminate (as a continuous truss in place of two or more simple spans) or the accurate analysis of an indeterminate structure in place of a crude estimate of the stresses (as in case of secondary stresses in bridges or wind stresses in building frames), the expense involved in making the necessary calculations will usually be altogether insignificant.

This is not meant to minimize the importance of simplifying the analysis of structures of whatever type in every legitimate way, for, rightly or wrongly, correct methods will be less widely adopted if they are tedious and complex than if they are short and simple.

155. Naturally Indeterminate Types.—The preceding discussion has been chiefly concerned with the merits and demerits of statically indeterminate construction as compared to alternate statically determinate forms. But it should be noted that there is a very large class of indeterminate structures, established as standard types in American practice, for which no corresponding determinate forms seem practically feasible. In such a class belong almost all reinforced concrete structures, the steel framework in office buildings and mill buildings, and also (if we take a somewhat broader definition of indeterminate construction) most heavy bridges which are fully or partially riveted. Difficulties in the joint details (especially the piling up of pin plates) has led to a very general adoption of a continuous upper chord in massive bridge trusses—even though they are nominally pin-connected trusses. In the case of concrete arches, three-hinged types are frequently seen in continental Europe and have occasionally been built elsewhere, but American practice thus far appears to favor overwhelmingly the hingeless type.

Though it is physically possible to introduce hinges into a steel building frame so as to convert it into a determinate structure, practically it would be quite difficult and undesirable. In a structure such as a reinforced concrete flat slab building frame or a multiple-arch dam, any modification to obviate the statical indetermination is altogether impracticable.

We may say that structures of this class are (in varying degrees) "naturally indeterminate," that is they are not rendered indeterminate by adding statically unnecessary supports or members, but from their

of interest to note Professor Turneaure's statement ("Modern Framed Structures," Part II, page 455) that "a good computer, after becoming familiar with the process, can make a complete analysis by joint loads of the secondary stresses in an ordinary truss in less than two days' time."

essential character special artifices would have to be used to render them determinate, and these appear in the main impracticable.

156. General Summary.—In so far as any definite conclusions can be deduced we may say that:

(1) Statical indetermination is never in itself a desideratum; certain inherent defects always accompany it which abstractly considered, place any indeterminate structure at a disadvantage.

(2) The essential defects of indeterminate construction are of widely varying practical importance; in some cases they are so serious as to completely bar such construction; in other cases they are of no practical consequence.

(3) When conditions are such as to minimize the importance of the essential defects, statically indeterminate types may show advantages in economy of material, stiffness, simplicity in manufacture and erection and such like over any other type practically feasible.

(4) Certain wide fields of construction as indicated in Art. 154 are practically preëmpted by forms that are essentially statically indeterminate and for which no alternative determinate type is practicable.

B. Historical Review

157. Early Period.—It is a surprising fact that structural engineering, though a very old practical art, is a very new science. On this point Professor H. Lorenz* remarks: "Despite the marked activity in construction of all civilized peoples in the ancient and medieval periods, there is no trace to be found in the literature of those times of any rational reflection on the strength of structural members or the fundamental properties of structural materials. Within the circle of constructive artisans, one was apparently satisfied with simple rules of thumb which were passed on from generation to generation, jealously guarded as secrets of the guild, and only rarely extended by new knowledge and experience. The architects in charge on the other hand, regarded themselves (even as to-day) as constructive artists; they seldom went beyond the application of the law of the lever (known since the time of Archimedes), in which they implicitly regarded the materials of construction as rigid bodies." This condition remained unchanged until the beginning of the 17th century, and it was not until about the middle of the 19th century that any systematic and comprehensive theory of structures was developed.

If we take one of the simplest, though one of the most important problems in structural mechanics, that of finding the stresses in a simple

* Technische Elastizitätslehre," pp. 644-5.

truss with smooth pin joints, we find the first definite step toward a solution in the work of the Dutch engineer, Simon Stevin (1548–1620) who appears to have understood the principles of composition and resolution of forces and to have made some primitive use of the force triangle. He investigated the problem of the loaded cord or rope—statically quite similar to the problem of the truss joint. Stevin's investigations were published in 1608. P. Varignon (1654–1722) the "founder of graphic statics" also investigated the loaded cord as well as other problems, enunciated the parallelogram law (apparently independently of Sir Isaac Newton) and pointed the way to many applications of the force polygon and string polygon. The principles developed by Varignon were applied to a variety of structural problems by the great French engineers of the first half of the 19th century, particularly Lamé, Clapeyron and Poncelet, but it does not appear that any marked advance in analytical or graphical method was made until after the middle of the period when a number of important discoveries followed in rapid succession. Before discussing these we may note that prior to 1850 the jointed truss was almost exclusively an American structure. A steady development in this type of construction had followed the Revolutionary War, and bridges up to 300-ft. span had been built. These were not built from rational designs, but in 1847 Squire Whipple, a prominent American engineer and inventor of the Whipple truss, published a remarkable treatise on "Bridge Building" in which he set forth for the first time a correct and tolerably complete theory of truss analysis and design. The methods he used are not the ones now followed, but this does not detract from the exceptional originality and thoroughness of his work.

In 1863 Prof. August Ritter* published his "Method of Sections" later to be so widely used, and indicated how all stresses might be analytically calculated by the principle of moments. In 1864 Prof. J. Clerk Maxwell published his work on "Reciprocal Figures and Diagrams of Forces" (the so-called "Maxwell stress diagram"); in 1866 Prof. Carl Culmann, of Zürich, the founder of modern graphics, published the first edition of his great treatise—"Die Graphische Statik." While much important work has been done by later scholars, these works definitely cleared up the general question of the rational analysis of the jointed frame.

157a. The history of the analysis of the simple beam runs quite parallel to that of the frame. The first speculations on the subject are attributed to Galileo (1564–1642). He investigated mathematically the strength of a cantilever beam of rectangular section loaded at the

* Continental authorities usually cite Ritter as the first to correctly analyze the stresses in a truss, but Whipple's prior claim seems clear.

end with a single concentration, arriving at a formula we now know to be quite erroneous. This is not surprising, since he "treated solids as inelastic, not being in possession of any law connecting the displacements produced with the forces producing them, or of any physical hypothesis capable of yielding such a law."* But the problem commonly known for 200 years as "Galileo's problem" marked the beginning of the modern theory of the stress-strain relations in elastic solids and it remained unsolved until 1820 when Claude Louis Marie Navier (1785–1836) distinguished French engineer and professor at the Ecole des Ponts et Chaussées presented to the French Academy a paper† giving a fairly full and sound treatment of the deflection and strength of beams. This was followed shortly by his memorable paper on the general theory of elasticity.‡

In 1826 Navier published the first edition of his "Leçons," § which not only contained the first adequate account¶ of what is frequently called the "common theory" of the flexure of beams, but also treated arches, suspension bridges, columns under eccentric loads and other technical problems. To Navier therefore belongs the double honor of developing the first general theory of elastic solids and also the first systematic treatment of the theory of structures.

During the period between Galileo and Navier many important developments were made, the most fundamental of which was the formulation of a law connecting elastic strain with the forces causing it. This was due to Robert Hooke (1635–1702), Professor of geometry at Gresham College, London, who arrived at the law bearing his name during the course of his investigations of steel springs to be used for clocks and watches. His discovery was made in 1660 but was not published until 1676 (as an anagram "ceiiinosssttuu" containing the letters of the Latin form of the law—("Ut tensio sic vis"). Hooke made no applications of his law to engineering problems, but in 1680 E. Mariotte (1620–1684) announced the same law (apparently quite independently) and applied it to Galileo's problem. His analysis appears correct for the simple cases treated.

Another important step in advance was the introduction of the physical notion of modulus of elasticity by Thomas Young (1773–1829).

* Love, "Mathematical Theory of Elasticity," page 2.

† Memoir sur la Flexion de verges élastiques courbes.

‡ "Mémoirs sur les lois d l'équilibre et du mouvement des corps solides élastiques."

§ "Résumé des leçons données a l'école des ponts et chaussées sur l'application de la Mécanique a l'établissement des constructions et des machines."

¶ A substantially correct theory of the flexure of beams (shearing effects entirely neglected) was proposed by Coulomb in 1776, though apparently not very fully elaborated.

There was further a vast amount of work done by eminent mathematicians and physicists on isolated problems in elasticity. Especially noteworthy were the studies of James Bernoulli (1654–1705) on the elastic curve of bent bars; of Daniel Bernoulli (1700–1782) and Leonhard Euler (1707–1783) on the same subject and on the vibration of beams and rods; of Euler and Lagrange (1736–1813) on the stability and strength of columns; of Mlle. Sophie Germain (1776–1831) on the vibration of plates; and of Coulomb (1736–1806) on bending and torsion.

158. *Middle Period.*—Following Navier's formulation of a general theory of elastic solids came a period of great activity and rapid development both in technical elasticity and the broader reaches of the subject as a branch of mathematical physics. With the latter we are not concerned here; among the important technical advances prior to 1860 we may note the memoir* of Poisson (1781–1840) published in 1829, containing the solution of some important plate problems and introducing the notion of transverse strain ("Poisson's ratio"); the statement by Clapeyron† of the theorem of equality between the internal work of deformation in an elastic solid and the work of the force producing it—a theorem used on the basis for many later investigations; the work of Lamè (1795–1820), one of the great pioneers in both the technical and more general science of elasticity, on cylinders and plates and elastic properties of iron (he also introduced the notion of the stress ellipsoid, and of curvilinear coördinates and wrote the first systematic treatise on the subject), and the general analysis of flexure, shear and torsion of any prismatic body by Barré de Saint-Venant (1797–1886), perhaps the greatest of elasticians. His work, culminating in a famous memoir‡ presented to the French Academy in 1855 may be said to have conclusively settled in all its practically important phases the "beam problem." Navier's solution was a satisfactory solution for all cases where flexure alone was the important action, and it is still the method used in most engineering applications. But Navier himself recognized that it was only applicable to deep, narrow beams. We may say then, roughly, that it was not until past the middle of the 19th century that simple truss action and simple beam action were fully understood, hence not until after this time that an adequate theory of structures could develop. Before outlining the development of the modern theory a few points regarding the earlier work seem worth noting.

* Mémoirs sur l'équilibre et le mouvements des corps élastiques."

† G. Lamè et E. Clapeyron,—"Sur l'équilibre intérieur des corps solides homogènes," Paris, 1833.

‡ Usually referred to as the "Memoir on Torsion," though containing a general treatment of the entire behavior of bars.

(*a*) During the period we have been considering no clear distinction was made between theory of elasticity and theory of structures. As we have just noted, the latter could not exist until the fundamental questions regarding elastic behavior were settled, and this led scientifically inclined engineers into the study of the theory of elasticity. The founder of this theory and two of the greatest contributors to its development, Navier, Lamè and de Saint-Venant were what would now be called professional structural engineers.

(*b*) As previously noted, prior to 1860 the jointed truss was little known or used outside America. Hence in one sense of the term (see pp. 341–2) all structures in common use in England and Europe were statically indeterminate internally. Doubtless due to this cause, no such emphasis on the distinction between determinate and indeterminate structures was made during the period just considered as has been in recent times. The question seems to have arisen chiefly in regard to fixed ended beams and arches and beams on several supports. As far back as Navier's "Leçons" at least, it was clearly realized that this problem was capable of rational solution by a consideration of the elastic behavior of the structure and in this way only.

(*c*) It is interesting to note in connection with the preceding that the order in which the basic structural problems were solved has no relation to their theoretical difficulty. The theory of the buckling of columns and the theory of arches and suspension bridges was developed before the theory of simple trusses, and many intricate problems, statically indeterminate in a high degree, regarding the stresses in plates, cylinders and the like were solved before the relatively simple analysis of continuous girders was perfected. It is historically quite inaccurate to regard the theory of indeterminate stresses as a modern development, a refinement, as it were, of the theory of simple structures.

(*d*) The earlier investigations of the stresses in such structures as continuous girders and arches were very intricate and laborious. However, simplifications and improvements were rapidly developed; Clapeyron published his treatment of continuous girders by the "three moment" theorem in 1857, vastly simplifying the whole subject, while Bresse * and Winkler† during the period from 1850 to 1865 presented very

* Bresse's first treatise—"Recherches analytiques sur la flexion et la résistance des pièces courbes" was published in 1854; most of the matter on structures was reproduced in his "Cours de mécanique appliquée" in 1859. It is worthy of note that certain recent French authorities insist that his general methods have never been improved upon for the treatment of beams and arches. On this point see Pigeaud, "Résistance des Matériaux," pp. VI-VII.

† "Formänderung und Festigkeit gekrümter Körper, 1856" and "Elasticität und Festigkeit," 1865.

thorough and practically usable analyses of curved beams and arches.

159. *Modern Period.*—The groundwork of the modern theory of indeterminate structures was laid during the period 1865–1880. In 1864 Maxwell (1830–1879)* published his analysis of a redundant framework by a method based on Clapeyron's theorem of the equality of the internal and external work of the actual loads on a structure. He also gave in this paper the law of reciprocal deflections. The treatment was brief and without any attempt to develop all the implications of the method or illustrate it by practical examples, and consequently it lay practically unnoticed for many years. In 1874 Mohr † (1838–1920), apparently quite without knowledge of Maxwell's works, gave a simpler and more comprehensive presentation, based on the principle of virtual work, of the same general method, together with examples of its varied application. The method is therefore widely known as the "Maxwell-Mohr" method. Several years prior to the preceding work Mohr had presented an epoch-making paper ‡ on the general representation of the elastic curve as a string polygon ("method of elastic weights.")

In 1879 Castigliano (1847–1884) published his treatise on the "Théorie de l'équilibre des systèmes élastiques" by the method of least work.§ This was a remarkably original and comprehensive treatise, covering a much wider range than the work of Maxwell and Mohr, and it had a very important influence on the development of the theory of indeterminate structures.

In 1879-80 Manderla presented his analysis of the secondary stresses in a truss with rigid joints. The unique feature of this solution was the use of the tangential angle at the member-ends as the unknown to be solved for rather than the moments or stresses direct.

With the preceding work the full basis for the modern theory of structures was laid. Other important work was of course done in this period; special mention should be made of Prof. Green's presentation of

* "On the Calculations of the Equilibrium and Stiffness of Frames." Philosophical Magazine, Vol. 27, 1864.

† "Beitrag zur Theorie des Fachwerks." Zeitschr. des Arch.- und Ing.-Vereins in Hannover, 1874-5.

‡ Beitrag zur Theorie der Holz- und eisen Konstriktionen. Zeitschr. des Arch.- und Ing.-Vereins in Hannover, 1868.

§ The method of least work, at least in a primitive form was used by Euler in his investigation of the elastic curve of beams and columns. D. Bernoulli suggested to him that the form of the true elastic line might be determined by making the total internal work a minimum. Also Ménabréa in a paper "Nouveau principe sur la distribution des tensions dans les systèmes élastiques," (Comptes Rendus, 1858), gave a definte statement of the principle as applied to trusses.

the method of moment areas (1872), of Williot's discovery of the construction bearing his name (1877), of the work of Winkler on the theory of arches (1868-9) and of Winkler and Asimont on secondary stresses (1880).

Since 1880 the development of the literature on theory of structures has been so vast that it is impossible here to do more than indicate a few of the more important contributions.

The full development of the Maxwell-Mohr theory in application to all structural problems has been largely due to the later works of Mohr himself and to Müller-Breslau (1851-1925) and A. Föppl (1854–1924) Müller-Breslau, Fränkel and others have also made a wide application of the principle of least work, Föppl, Henneberg and Müller-Breslau* developed the theory of space frameworks, determinate and indeterminate. W. Ritter, following Culmann and Mohr, developed graphical methods of treatment for a very wide variety of statically indeterminate problems, among others the solution by the ellipse of elasticity which has recently been given considerable attention in American literature. A very elegant graphical solution of the continuous girder problem by the method of "characteristic points" was presented by Claxton Fidler (Practical Treatise of Bridge Construction, 1887), and this was elaborated and extended by A. Ostenfeld† to include the general case of angular or linear yield of supports. Engesser and Mohr have contributed largely to the later development of secondary stress theory; particular mention should be made of Mohr's method of solution by "slope-deflections" first proposed in 1892.‡

J. Melan has been one of the leaders in developing the modern theory of suspension bridges.§ Since the failure of the Quebec bridge in 1907 brought into prominence the question of the behavior of large built-up columns a new theory has been presented for the action of such members, largely due to Müller-Breslau ¶ and Engesser.

Particular attention of late has been given to the very difficult subjects of elastic stability, stresses in medium-thick plates and stresses in domes and multiple arch-dams. Though these fall within the scope of statically indeterminate stresses, properly speaking, they are not ordinarily so included in standard treatises, since they require lengthy

*A full citation of the published articles is too lengthy to insert here; those interested are referred to the very full bibliography in Mehrtens, "Statik und Festigkeitslehre," III-2nd Hälfte, pp. 258-266.

† "Graphische Behandlung der continuerlichen Trager, etc." Zeitschr. für Arch.- und Ingenieurwesen, 1905 and 1908.

‡ See note on page 156 regarding the development of the slope-deflection method.

§ See bibliography, page 360.

¶ Neuer Methoden der Festigkeitslehre, Abschnitt V.

and special treatment. A large literature in this line has developed in the last few years, but no attempt will be made even to outline it here. Some basic references may be found in the appended bibliography.

C. Bibliography

The following brief bibliography is intended to give the student (1) a fairly complete list of the more recent books in English which treat the general subject of indeterminate stresses, or important departments of it; (2) a list of a few representative foreign treatises, and (3) a short list of papers and monographs in English which cover a different field, or a special field more completely than do the general treatises. Brief descriptive comment is appended in some cases.

The literature bearing on the theory of statically indeterminate structures is now so voluminous that anything like a comprehensive bibliography would be much too bulky to insert in this book. It is hoped that the small list given will serve to introduce the student to the larger field. Some of the references named contain rather full bibliographies.

Among the general treatises on statically indeterminate structures may be mentioned:

Andrews, Ewart C. Translation of *Théorie de l'équilibre des systèmes élastiques*, by A. Castigliano, under the title, Stresses in Elastic Structures. London. Scott & Greenwood. 1919.

Though nearly 50 years old, this remarkable treatise still has much more than mere historic interest, and is well worth careful study.

Church, I. P. Mechanics of Internal Work. New York. John Wiley & Sons. 1910.

A clear-cut exposition of fundamental principles.

Hiroi, Isami. Statically Indeterminate Stresses. New York. Van Nostrand. 1905.

Brief, clear treatment (exclusively by method of least work) the leading types of indeterminate structures and of secondary stresses.

Hool, George A., and **Kinne, W. S.** Structural Engineers Handbook Library, volumes entitled, Structural Members and their Connections. Stresses in Framed Structures. Movable and Long-Span Bridges. New York. McGraw-Hill. 1923.

No one of these volumes is primarily devoted to the subject of indeterminate stresses, but taken as a whole they cover the subject rather thoroughly.

The first named volume contains a very thorough and excellent treatment of beam deflections by a variety of methods and of restrained and continuous beams;

the second volume treats (among other things) the subject of truss deflections, redundant members, secondary stresses and rigid frames; the third named volume treats at considerable length the theory and practice of continuous and swing spans, arches and suspension bridges.

Hudson, Clarence W. Deflections and Statically Indeterminate Stresses. New York. John Wiley & Sons. 1912.

Thorough treatment of deflections in general and of continuous girders, swing bridges and arches. Secondary stresses and suspension bridges and rigid frames are not treated. Method of consistent deflections and method of least work are both used.

Johnson, J. B., Bryan, C. W., and **Turneaure, F. E.** Modern Framed Structures, Part II. Statically Indeterminate Structures and Secondary Stresses. New York. John Wiley & Sons. 1911.

Taken in connection with the last chapter of Part I of the same series, which chapter treats of deflections and the elementary applications to indeterminate problems, this volume probably offers the best and most comprehensive treatment of statically indeterminate stresses in the English language. All practically important types of structures are treated. The treatment of secondary stresses and suspension bridges is especially full and detailed. The method of consistent distortions is the basic method followed, though the method of least work is illustrated.

Merriam, Mansfield, and **Jacoby, Henry S.** Bridges and Roofs, Part IV. Higher Structures. New York. John Wiley & Sons. 1907.

Full treatment of continuous girders, swing bridges and arches and an elementary treatment of suspension bridges.

Molitor, D. A. Kinetic Theory of Engineering Structures. New York. McGraw-Hill. 1911.

Contains full treatment of fundamental theory. The method of approach is quite different from most other treatises in English, being largely modeled after the European method.

Among the more important special treatises may be named:

Balet, Joseph W. Analysis of the Elastic Arch. New York. Eng. News Publishing Co. 1908.

Very thorough treatment of the standard theory of arches of all types and presents also a comprehensive method for approximate solution of arches (by means of the reaction locus) without the use of statically indeterminate analysis.

Burr, William H. Suspension Bridges. New York. John Wiley & Sons. 1913

Very full treatment of suspension bridge theory; also considerable space devoted to the theory of arches.

Grimm, C. R. Secondary Stresses. New York. John Wiley & Sons. 1908.

Full account of the various methods of solution of the secondary stress problem, illustrated by many examples.

Hool, George A. Reinforced Concrete Construction, Vol. III. Bridges and Culverts—Part I—Arch Bridges. New York. McGraw-Hill. 1916.

Full treatment of reinforced concrete arch by the standard method and also by the method of the ellipse of elasticity. Also treats problem of multiple arch bridge with elastic piers.

Howe, Malverd A. A Treatise on Arches. New York. John Wiley & Sons. 1897.

Detailed mathematical treatment supplemented by tables. Also chapters on comparative design, tests, etc.

Steinman, D. B. Translation of J. Melan's Arches and Suspension Bridges. Chicago. Myron C. Clarke. 1913.

Very thorough treatment of arches and suspension systems by one of the foremost living authorities.

Steinman, D. B. Translation of J. Melan's The Reinforced Concrete Arch. New York. John Wiley & Sons. 1915.

Very complete treatment supplemented by tables, graphs and examples.

Steinman, D. B. A Practical Treatise on Suspension Bridges. New York. John Wiley & Sons. 1922.

Brief and simplified treatment of the theory; three-fourths of the book is devoted to treatment of design and construction.

A number of works which are not treatises on indeterminate structures primarily have important sections devoted to the subject. Among these may be named:

Burr, Wm. H., and **Falk, Myron B.** Design of Metallic Bridges. New York. John Wiley & Sons. 1905. Chapters VII and VIII.

Ellis, Charles A. Theory of Framed Structures. Chapters V to VIII. New York. McGraw-Hill. 1922.

Green, Charles E. Trusses and Arches. Chapters VII to X. Part II; practically all of Part III. New York. John Wiley & Sons. 1893-94.

Ketchum, Milo S. Design of Steel Mill Buildings. Part II,—Chapters XIV to XXII. New York. McGraw-Hill. 1922.

Morley, Arthur. Theory of Structures. Chapters VI, VII, XIV, XV, XVIII. London. Longmans, Green. 1918.

Spofford, Charles M. Theory of Structures. Chapters XIV to XVII. New York. McGraw-Hill. 1915.

The literature on indeterminate stresses in continental Europe is wide and varied, and by far the greater bulk of it is in the German language. Among the modern comprehensive treatises covering the entire field may be mentioned:

Mehrtens, G. C. Statik und Festigkeitslehre. Leipsig. Wilhelm Englemann. 1912.

This work is in three volumes; parts of Vols. I and II and nearly all of Vol. III are devoted to deflections and indeterminate stresses, and the treatment is very full and detailed.

Müller-Breslau, H. F. B. Die Graphische Statik der Baukonstruktionen. Stuttgart. Alfred Kroner. 1923-25.

This work is in three volumes and practically the whole of the last two are devoted to statically indeterminate stress—some 1200 pages. The treatment is probably the most complete to be found anywhere and the work is perhaps the leading international reference book on the subject.

Ritter, W. Anwendungen der Graphischen Statik, nach C. Culmann. Zürich. A. Raustein. 1900-1906.

This is a four-volume work of which the 3d and 4th volumes are devoted to indeterminate problems. The whole field is covered and graphic or semi-graphic methods predominate in the treatment. This is the leading reference work for graphic methods.

There does not appear to be any recent treatise in French covering the theory of indeterminate stresses in so comprehensive a manner as the German books cited, though the monumental treatise of M. Maurice Levy, "La Statique Graphique et ses applications aux constructions." (Paris, 1874, 2d edition, 1888) remains one of the great reference works. Among the recent books treating indeterminate stresses may be named:

Flamard, Ernest. Calcul des systèmes élastiques de la construction. Paris. Gauthier-Villars. 1918.

A small book containing a very thorough mathematical treatment of the fundamental "work theorems" and their applications to continuous and restrained beams, trusses and arches.

Pigeaud, Gaston. Résistance des Matériaux. Paris. Gauthier-Villars. 1923.

This large volume is, as the name indicates, primarily a treatise on mechanics of materials, but it contains several chapters devoted to the subject of indeterminate structures—continuous girders, arches and suspension bridges.

Mention should be made of a very comprehensive treatise in Danish covering the entire field of indeterminate construction:

Ostenfeld, A. Teknisk Statik, Vol. II. Copenhagen. 1913.

While no attempt will be made here to list all the important papers and monographs on indeterminate structures that have appeared in America in recent years, it may not be out of place to note the following for the reason that they treat certain important departments or phases of the theory not ordinarily found in text and reference books:

Von Abo, C. V. Secondary Stresses in Bridges. Proc. A. S. C. E. Feb., 1925.

This paper is devoted to a detailed critical comparison of the various methods of attack on the secondary stress problem. Together with the discussion it constitutes by far the most complete study of this important topic that has ever been made.

Beggs, George E. Mechanical Solution of Statically Indeterminate Structures by Paper Models and Special Gages. Proc. A. C. I., Vols. XVIII and XIX.

This gives the fundamental theory, illustrated by many applications of Prof. Beggs' unique method of solving indeterminate problems by means of mechanical models.

Janni, A. C. The Design of Multiple Arch Systems. Proc. A. S. C. E. Aug. 1924.

Perhaps the most complete exposition in English of the analysis of this important problem by means of the ellipse of elasticity.

Wilson, W. M., and Maney, George A. Wind Stresses in Office Buildings. Bulletin No. 80, Univ. of Ill. Expt. Station.

The most thorough and complete treatment of the problem that has so far appeared. It contains a full exposition of the exact solution by the slope-deflection method, a critical comparison of various approximate methods, and a fully worked out example of a 20-story building.

It has been noted in the historical summary that there are a number of problems in the theory of structures which are actually problems in statically indeterminate stresses, but which are not amenable to the methods analysis ordinarily included under this head. In this group we may include plate and dome action, elastic stability, and problems regarding the exact local distribution of stress. These problems usually require a more exact formulation of the stress-strain relations within an elastic solid than is necessary for most structural problems. It would appear that this field is becoming of increasing importance to

structural engineers, and for the benefit of those interested in studying up the subject a few references are noted here.

On the general subject of technical elasticity perhaps the best English work is:

Prescott, John. Applied Elasticity. London. Longmans, Green. 1924.

In German two books are available which are particularly well suited to the needs of the engineer.

Föppl, A. and **O.** Drang und Zwang. 2 Volumes. Munich and Berlin. R. Oldenburg. 1920-24.

Lorenz, H. Technische Elastizitätslehre. Munich and Berlin. R. Oldenburg. 1913.

Attention may be called to the following monographs on special problems in the field of technical elasticity which are of interest to structural engineers:

Salmon, E. H. Columns. London. Henry Frowde and Hodder & Stoughton. 1921.

Critical review of the theory of columns and of test data. Contains an exhaustive bibliography.

Smith, B. A. Arched Dams. Proc. A. S. C. E., March, 1920.

Probably the most thorough mathematical treatment of this important problem that has thus far appeared.

Westergaard, H. M. and **Slater, W. A.** Moments and Stresses in Slabs. Proc. A. C. I., Vol. XVII, 1921.

By far the most complete treatment of this subject (mathematical analysis and review of test data) that has appeared in English. Contains a full bibliography

Westergaard, H. M. Buckling of Elastic Structures. Proc. A. S. C. E., Nov., 1921.

Comprehensive mathematical treatment of the subject of elastic stability so far as it affects engineering structures. Very full bibliography.

INDEX

Date Due

Aug18 '43			
S AUG	7 1967		